MAT

2041

# MAT 2041 Linear Algebra and Applications
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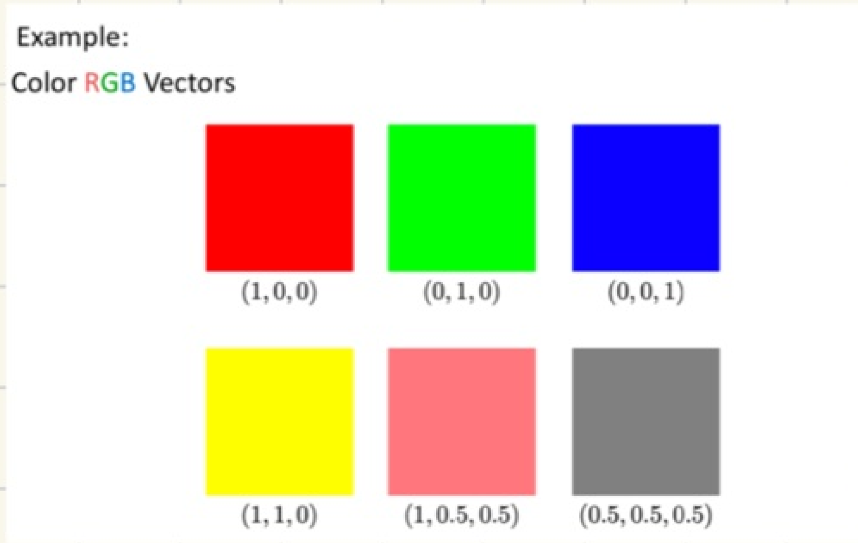
## I. Introduction to Vectors

- **Vectors**




Def. Column vectors

$\triangle$ $\vec{v} = [v_1, v_2, \cdots v_i, v_n]^T \overset{\text{Transpose}}{=} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \equiv (v_1, v_2, \cdots, v_n)$.

e.g. Zero vector $\vec{0} = (0, 0, \cdots, 0)$.
A list data structure in computer algorithms
A time series (a sequence of data points) in data science problems.

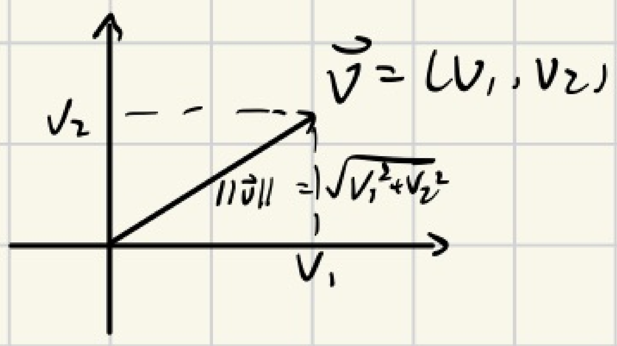
Operations
$\vec{v} = (v_1, v_2, \cdots v_n)$. $\vec{w} = (w_1, w_2, \cdots w_n)$.

① $\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)$. Vector Addition.

② $c\vec{v} = (cv_1, cv_2, \cdots, cv_n)$. Vector Multiplication

☆③ $c\vec{v} + d\vec{w} = (cv_1 + dw_1, cv_2 + dw_2, \cdots, cv_n + dw_n)$ Linear Combination

- **Vector Norm** ("Geometric Length")




Consider a n-dimensional vector

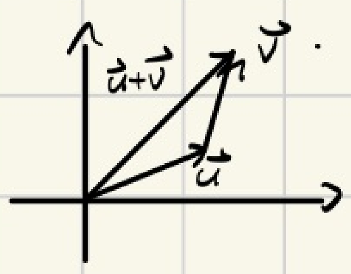
Def. ($\ell_2$-norm)

Let $\vec{v} = (v_1, v_2, \cdots, v_n)$ be a n-length vector. The $\ell_2$-norm of $\vec{v}$, denoted by $\|\vec{v}\|_2$ is defined as
$\|\vec{v}\|_2 = (v_1^2 + v_2^2 + \cdots + v_n^2)^{\frac{1}{2}}$.

(We often abbreviate $\|\vec{v}\|_2$ as $\|\vec{v}\|$).

Properties:

$\triangle$
① $\|\vec{v}\| \geq 0$, $\|\vec{v}\|$ iff $\vec{v} = 0$   (iff: if and only if)
② $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$
③ $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$




Triangle inequality.

Unit vector

Def. A vector $\vec{v}$ is called a unit vector if $\|\vec{v}\| = 1$.

For any non-zero vector $\vec{v}$, $\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector.

e.g. $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

- **Inner Product** (Dot Product)

Def. A dot product between 2 vectors (of the same size) $\vec{v} = (v_1, v_2, \cdots, v_n)$, $\vec{w} = (w_1, w_2, \cdots, w_n)$ is defined as $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w} = \langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} v_i w_i$.

Recommend writing as $\langle \vec{v}, \vec{w} \rangle$ to avoid confusion.

Properties of Inner Products.

① $\langle a\vec{v} + b\vec{u}, \vec{w} \rangle = a\langle \vec{v}, \vec{w} \rangle + b\langle \vec{u}, \vec{w} \rangle$ Linearity

② $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$ Symmetry

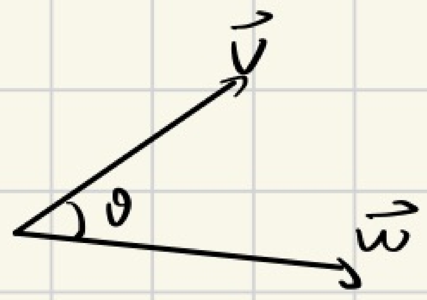
③ $\langle \vec{v}, \vec{v} \rangle = \|\vec{v}\|^2 \geq 0$. Positivety.

$\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = \vec{0}$

· Properties of Inner Product and Norm

1. Cosine Similarity

$$\cos\theta = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \cdot \|\vec{w}\|}$$




2. Cauchy-Schwarz Inequality

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \cdot \|\vec{w}\|.$$

3. Pythagoras Law

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v} - \vec{w}\|^2 \quad \text{iff} \quad \langle \vec{v}, \vec{w} \rangle = 0.$$

## II. System of Linear Equations

· Linear Equations

Def. A linear equation is the equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

where $a_i, b \in \mathbb{R}$. $x_i$ are variables.

Vector form: $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\langle \vec{a}, \vec{x} \rangle = b$.

e.g. $-x_1 + 4x_2 = 2x_2 + 3x_3$.

· System of LE

Def. An $m \times n$ system of linear equations is a collection of $m$ linear equations and $n$ variables.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$ Scalar Form

where $a_{ij}, b_i \in \mathbb{R}$. $x_i$ are variables.

Row-vector Form (Unknown vector satisfies n linear equations simultaneously).

$(x_1, x_2) \cdot (a_{11}, a_{12}) = b_1$

$(x_1, x_2) \cdot (a_{21}, a_{22}) = b_2$

Column-vector form (Unknown combination of columns produces vector $\vec{b}$).

$$x_1 \cdot \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + x_2 \cdot \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Matrix form (Given matrix times unknown vector produces $\vec{b}$).

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

## III. Matrix

### Definition

An $m\times n$ matrix $A$ is a rectangular array of numbers with $m$ rows and $n$ columns in the form:

$$\triangleright\ A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} =: (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}.$$

where all $a_{ij}$ are scalars. P.S. Matrix = row vector of column vectors.
= column vector of row vectors

Dimension:
For an $m\times n$ matrix, the dimension of $A$ is $m\times n$.
For an $m\times 1$ vector, the dimension of the vector is $m$, or $m\times 1$.

### Notation

① For a matrix $A$, $a_{ij}$ is called the $(i,j)$-th entry (element) of $A$.
② Matrices are denoted by $A, B, C, \ldots$
③ When $m=n$, $A$ is called a square matrix. Otherwise, it's called a rectangular matrix.
④ When all entries are zeros, $A$ is called a zero matrix. (similar to zero vector).
⑤ When $m=1$, $A$ is a row vector.
When $n=1$, $A$ is a column vector.

$\triangleright$ ⑥ Column of a matrix: $\vec{a_j}=\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$

Row of a matrix: $a^{(i)}=[a_{i1}, \cdots a_{in}]$.

### Matrix-vector Product

Def. Let $\vec{u}=\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $A$ be an $m\times n$ matrix $A=(a_{ij})$.

The matrix-vector product $A\vec{u}$ is an $m\times 1$ column vector:

$$A\vec{u}=\begin{bmatrix} \sum_{j=1}^{n} a_{1j}\cdot u_j \\ \sum_{j=1}^{n} a_{2j}\cdot u_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}\cdot u_j \end{bmatrix}$$

Remark: Dimension checking $\underset{m\times n}{A}\overset{n\times 1}{\vec{u}}=\overset{m\times 1}{\vec{v}}$

Properties:
① Row-form of matrix-vector product
Let $A$ be an $m\times n$ matrix and $\vec{b}$ an $n\times 1$ column vector. Suppose $A=\begin{bmatrix} \vec{a}^{(1)} \\ \vec{a}^{(2)} \\ \vdots \\ \vec{a}^{(m)} \end{bmatrix}$ then

$$A\vec{b}=\begin{bmatrix} \vec{a}^{(1)}\cdot\vec{b} \\ \vec{a}^{(2)}\cdot\vec{b} \\ \vdots \\ \vec{a}^{(m)}\cdot\vec{b} \end{bmatrix}$$

② Column-form of matrix-vector product.
Let $A$ be an $m\times n$ matrix and $\vec{w}$ be an $n\times 1$ column vector. Suppose $A=[\vec{a_1}, \vec{a_2}, \cdots, \vec{a_n}]$, then
$A\vec{w}=w_1\vec{a_1}+w_2\vec{a_2}+\cdots+w_n\vec{a_n}$.

# IV. Elimination

- ## Motivation

How to solve n by n system?

1. Eliminate one variable by subtracting one equation from others
2. Solve the remaining (n-1) by (n-1) system.
3. Continue the process until getting 1 variable and 1 equation.

- ## Coefficient Matrix & Augmented Matrix

Def. Given a linear equation,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m.$$

The coefficient matrix of the system is an $m \times n$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} =: (a_{ij})_{m\times n}$$

The corresponding augmented matrix is:

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

- ## Overdetermined, Underdetermined and Square

An $m \times n$ system of linear equations is

1° overdetermined system if $m > n$. (tall)
2° underdetermined system if $m < n$. (wide).
3° square system if $m = n$. (square).

- ## Special Matrices

Lower Triangular Matrix

Def. A square matrix $(L_{ij})_{n\times n}$ where $L_{ij} = 0$. for any $1 \le i < j \le n$.

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & \cdots & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}$$

Upper Triangular Matrix

Def. A square matrix $(U_{ij})_{n\times n}$ where $U_{ij} = 0$, for any $1 \le j < i \le n$.

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Diagonal Matrix

Def. A square matrix $(D_{ij})_{n\times n}$ where $D_{ij} = 0$, $\forall i \ne j$.

$$\Rightarrow \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

($d_{ii}$ can be 0!)

## Gaussian Elimination

Step 1: Forward Elimination → triangular matrix

Perform elementary row operations and try to get an upper triangular matrix

Step 2: Backward Substitution → diagonal matrix

Perform elementary row operations and try to get a diagonal matrix.

Assumption 1: At each iteration of the forward elimination, the pivot is nonzero. (may not hold for some problems)

↳ Claim 1: Under A1, we can get a diagonal matrix at the of step 2.

Corollary 1: Under A1, the system has a unique solution.

· **Allowable Operations on Rows**

Given $A = \begin{bmatrix} a^{(1)} \\ a^{(2)} \\ \vdots \\ a^{(m)} \end{bmatrix}$

① [Multiplication] Multiply a row by a non-zero scalar.

$a^{(i)} \longrightarrow \lambda a^{(i)}$.

② [Addition] Add to one row a scalar multiple of another.

$a^{(i)} \longrightarrow a^{(i)} + \lambda a^{(j)}$.

③ [Interchange] Swap the positions of two rows.

$\begin{bmatrix} a^{(1)} \\ \vdots \\ a^{(i)} \\ \vdots \\ a^{(j)} \\ \vdots \\ a^{(m)} \end{bmatrix} \longrightarrow \begin{bmatrix} a^{(1)} \\ \vdots \\ a^{(j)} \\ \vdots \\ a^{(i)} \\ \vdots \\ a^{(m)} \end{bmatrix}$ (swap i-th row & j-th row).

# V. Matrix Multiplication

· **Definition**

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times k}$. The product $C = AB$ is an $m\times k$ matrix where:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

AB exists iff A's column # = B's row # Dimension Match.

Q: How many multiplications?

A: $m \times n \times k$.

· **Properties**

$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$

Let $A, B, C$ be 3 matrices, $I \in \mathbb{R}^{n\times n}$ is an identity matrix, $\alpha \in \mathbb{R}$, we have:

zero matrix

0. $AI = A$. $A\,0 = 0$
1. $A(B+C) = AB + AC$, $A \in \mathbb{R}^{m\times n}$, $B, C \in \mathbb{R}^{n\times k}$ — Left Distributive
2. $(B+C)A = BA + CA$, $B, C \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{n\times k}$ — Right Distributive
3. $\alpha(AB) = (\alpha A)B = A(\alpha B)$. $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times k}$. — Scalar Associate
4. $(AB)C = A(BC)$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times p}$. — Associative

Warning: AB & BA?

1. AB exists doesn't imply that BA exists.
2. Even if both AB & BA exists, they are generally not equal. ($AB \neq BA$).

## · Transpose

Def. Let $A=(a_{ij})_{m\times n}$, then the transpose of $A$ is the matrix $B=(b_{ji})_{n\times m}$, where $b_{ji}=a_{ij}$ $(i=1,2,\cdots m,\ j=1,2,\cdots n)$.

Notation: $B=A^T$

e.g. $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$, $A^T=\begin{bmatrix}1&3\\2&4\end{bmatrix}$ $A=\begin{bmatrix}1&0&3\\5&-1&0\end{bmatrix}$, $A^T=\begin{bmatrix}1&5\\0&-1\\3&0\end{bmatrix}$

$A=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $A^T=[1,2,3]$

Properties:

Let $A,B\in R^{m\times n}$, $\alpha\in R$, then

1. $(A+B)^T=A^T+B^T$
2. $(\alpha A)^T=\alpha A^T$
3. $(A^T)^T=A$
4. $(AB)^T=B^TA^T\neq A^TB^T$ ☆

## · Symmetric Matrix

Def. If a matrix $A\in R^{n\times n}$ satifies $A=A^T$, we call $A$ is symmetric.

e.g. $A=\begin{bmatrix}1&2&-1\\2&0&0\\-1&0&3\end{bmatrix}$.

## · Vector Outer Product

Def. Let $\vec{u}\in R^{m\times 1}$ and $\vec{v}\in R^{n\times 1}$ are two column vectors. The outer product of $\vec{u}$ and $\vec{v}$ is:

$$\vec{u}\vec{v}^T=\begin{bmatrix}u_1\\u_2\\\vdots\\u_m\end{bmatrix}[v_1\ v_2\cdots v_n]=\begin{bmatrix}u_1v_1&u_1v_2&\cdots&u_1v_n\\u_2v_1&u_2v_2&\cdots&u_2v_n\\\vdots&\vdots&\ddots&\vdots\\u_mv_1&u_mv_2&\cdots&u_mv_n\end{bmatrix}$$

PS: The outer product of $\vec{u}$ and $\vec{v}$ is $\vec{u}\vec{v}^T$ => Matrix
The inner product of $\vec{u}$ and $\vec{v}$ is $\vec{u}^T\vec{v}$ => Scalar

## · Block Partition

Def. The matrix $A=\begin{bmatrix}A_{11}&\cdots&A_{1t}\\\vdots&\ddots&\vdots\\A_{s1}&\cdots&A_{st}\end{bmatrix}$ is a partition of matrix with $s\times t$ blocks if $A_{ij}$:

① For each fixed $i$, the number of rows of all $A_{ij}$ are equal.
② For each fixed $j$, the number of columns of all $A_{ij}$ are equal.

The matrix $A_{ij}$ is called the $(i,j)$-block of $A$.
We say $A$ has $s$ block-rows and $t$ block-columns.

Multiplication of Partitioned Matrices

Rule 1: Block structure match.
The number of block-columns in $A$ must equal the number of block-rows in $B$.
Then can treat each block as an "entry", for standard matrix multiplication.

Rule 2: Block size match
When multiplying two blocks, the dimensions of the blocks must match.

# VI. Breakdown Cases & REF, RREF

## · Zeros in GJ-E

Type 1: No solution

e.g. $\begin{cases} x_1+x_2=2 \\ 2x_1+2x_2=5 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right]$ ?

Type 2: Infinite solutions

e.g. $\begin{cases} x_1+x_2=2 \\ 2x_1+2x_2=4 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$ ?

Type 3: 1 solution (0 in diagonal).

e.g. $\begin{cases} x_2=2 \\ 2x_1+2x_2=4 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{cases} x_1=0 \\ x_2=2. \end{cases}$

## · Solution Test

Def. A solution of an $m\times n$ linear system with variables $(x_1, x_2, \cdots, x_n)$ is a vector such that if we let $x_i = s_i$ for all $i=1,2,\cdots,n$, the $m$ equations hold simultaneously.
The solution set of an $m\times n$ linear system is a set that contains all solutions.

e.g. $\begin{cases} 2x_1+3x_2=3 \\ x_1-x_2=4 \end{cases} \Rightarrow$ The solution set is $\{(3,-1)\}$.

$\begin{cases} 2x_1+3x_2=3 \\ 4x_1+6x_2=6 \end{cases} \Rightarrow$ The solution set is $\{(t, \frac{3-2t}{3}) | t\in\mathbb{R}\}$.

## · REF & RREF

Def. A matrix is in row echelon form (REF) if

1. Leading 1s: The first entry of each non-zero row is 1. (called "pivot").
2. Leading 1s Moving Right: If a row has a leading 1, the leading 1 in the next row appears in a later column.
3. Zero Rows at the Bottom: If there is a row whose entries are all zero, then all rows below it have all zero entries.

And is reduced row echelon form (RREF) if

4. Zeros in the column of the leading 1: Each leading 1 is the only nonzero entry in its column (the corresponding column is called pivot column)

e.g. REF: $\begin{bmatrix} \underline{1} & 2 & 2 & 3 & 1 \\ 0 & \underline{1} & 5 & 4 & 5 \\ 0 & 0 & 0 & \underline{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ RREF: $\begin{bmatrix} \underline{1} & 0 & 2 & 0 & 1 \\ 0 & \underline{1} & 5 & 0 & 5 \\ 0 & 0 & 0 & \underline{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Property of RREF:

If a matrix $A$ is in RREF, then exchanging columns of $A$ can lead to the form

$$\begin{bmatrix} I_k & F \\ 0 & 0 \end{bmatrix}$$

where $I_k$ is the $k\times k$ identity matrix.

· Number of Solutions for Square Systems

Problem: Solve $n \times n$ linear system of equations.
Preparation: Write the linear system as argumented matrix form.
Step 1:
Perform elementary row operations to get an upper triangular matrix.
Step 2:
Case 1: All diagonal entries are nonzero (i.e. $n$ pivots). Unique solution.
Then perform "row multiplication and addition" to get an identity matrix.
Case 2: Some diagonal entries are zero (i.e. less than $n$ pivots):
At the end of elimination, if:
① exists $0 = *$ (nonzero number). No solution.
② besides $x_i + * = *$, all are $0 = 0$. $\infty$ many solutions

## VII. Matrix Inverse

· Matrix Inverse

Def. Suppose $A \in \mathbb{R}^{n \times n}$. If a matrix $B \in \mathbb{R}^{n \times n}$ satisfies $AB = BA = I_n$ (nxn union matrix.), then we say $B$ is the inverse of $A$.
Denoted as $B = A^{-1}$.

Suppose $A \in \mathbb{R}^{n \times n}$. If there exists $B$ such that $B = A^{-1}$, then we say $A$ is invertible.
If $A$ is invertible, then we say $A$ is nonsingular.
If $A$ is not invertible, then we say $A$ is singular.

e.g. Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$.

then $AB = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thus, $A^{-1} = B = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$.

Theorem: Matrix inverse is unique. Uniqueness
Suppose the square matrix $A$ has an inverse. Then $A^{-1}$ is unique.

Theorem: If $A$ is invertible, then the linear system $Ax = b$ has a unique solution $x = A^{-1}b$.

· Matrix Inverse of Special Matrices

1. Diagonal Matrix

A diagonal matrix $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is inversible if $a_{ii} \neq 0$ $(i = 1, 2, \cdots, n)$.

and the inverse is $\begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$

2. Triangular Matrix

① A triangular matrix is invertible iff the diagonal entries are invertible.
② The inverse of an upper (lower) triangular matrix (if exists) is upper (lower) triangular matrix.

· Inverse of a 2×2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$.

And $A^{-1}$ exists iff $\det(A) \neq 0$.

· Inverse of Products

Theorem: Matrix Inverse of Matrices Product
Suppose $A$ and $B$ are invertible matrices of size $n$, then $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

e.g. $A_1, A_2, \cdots A_k$ are real invertible matrices.
Then $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}$.

## VIII. Elementary Matrix

· Elementary Matrix

Def. The matrices corresponding to a single elementary row operation are called elementary matrices.

o For a given matrix $A$, performing elementary row operation for $A$ is equivalent to premultiplying $A$ by the corresponding elementary matrix.

· Inverse of Elementary Matrix

Proposition:
1. $E_{R_iR_j}^{-1} = E_{R_iR_j}$, corresponding to the reverse row operation 1: $R_i \leftrightarrow R_j$.
2. $E_{\alpha R_i}^{-1} = E_{\frac{1}{\alpha}R_i}$ $(\alpha \neq 0)$. corresponding to the reverse row operation 2: $R_i \to \frac{1}{\alpha}R_i$.
3. $E_{\beta R_i + R_j}^{-1} = E_{-\beta R_i + R_j}$, corresponding to the reverse row operation 3: $R_j \to -\beta R_i + R_j$.

Remark: The inverse of the elementary matrices corresponding to the reverse row operations and belong to the same type of elementary matrices.

## IX. LU Decomposition

· n by n Good Case

Given a coefficient matrix $A$ (square matrix).
We have Gaussian elimination:

$A \to A_1 \to A_2 \cdots \to U$

$\Rightarrow E_k \cdot E_{k-1} \cdots E_1 \cdot A = U$

$\Rightarrow A = (E_k \cdot E_{k-1} \cdots E_1)^{-1} \cdot U \quad \Rightarrow \quad A = LU$

$L = (\prod E_i)^{-1}$

- **Algorithm of Computation**

Step 1: Forward elimination.
Run forward elimination, till get upper triangular matrix $U$.
IF row exchange needed:
STOP and report: the algorithm fails to generate LU decomposition.
ELSE (no row exchange) Go to Step 2.

Step 2: Record elementary matrices
Record $E_1, E_2, \cdots, E_k$ in Step 1.

Step 3: Compute $L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.

Step 4: Conclusion
$A$ has an LU decomposition $A = LU$.

- **PLU Decomposition**

Theorem: Any square matrix $A$ can be written as
$$PA = LU$$
where $L$ is lower triangular, $U$ is upper triangular, $P$ is a certain permutation matrix.

- **Permutation Matrix**

Def. Let $e_i$ be the $i^{th}$ row of the identity matrix $I$.

e.g. If $\pi = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, then $P_\pi \begin{bmatrix} e_1 \\ e_3 \\ e_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

A permutation matrix is a square matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere.

The product of row exchange matrices
$$P = E_{R_{i_1} R_{j_1}} E_{R_{i_2} R_{j_2}} \cdots E_{R_{i_k} R_{j_k}}.$$
is a permutation matrix

## X. Matrix Inverse Continue

- **When is $A$ invertible?**

Ans: $A$ is invertible iff $A$ has $n$ pivots (Assuming $A$ is $n$ by $n$ matrix).

- **Left Inverse and Right Inverse**

Def. Let $A$ and $B$ be matrices. If $BA = I$ (AB), where $I$ is the identity matrix, then $B$ is called the left (right) inverse of $A$.

Lemma: If $MA = I_n$, (and $M$ is invertible) (not necessary), then $A^{-1} = M$.

- **How to express $A^{-1}$?**

We have G-J-E (case of $n$ pivots): $A \to A_1 \to A_2 \to \cdots \to U \to \cdots B_1 \to \cdots \to I_n$.
Noted $U \to E_p \to E_{p-1} \to \cdots \to E_1 \to I_n$. And we can get $A^{-1} = E_p E_{p-1} \cdots E_2 E_1$.
Claim: A matrix is invertible if it can be written as the product of elementary matrices.

· How to compute $A^{-1}$?

Algorithm 1:

Step 1: Forward elimination (with column exchange)
Run forward elimination (with column exchange), till get upper triangular matrix $U$.
If $U$ contains zero diagonal entry:
STOP and report: No inverse.
ELSE Go to Step 2.

Step 2: Run backward substitution, till get identity matrix $I_n$.

Step 3: Record elementary matrices $E_1, E_2, \cdots, E_k$ in Step 1.
Record elementary matrices $E_{k+1}, E_{k+2}, \cdots, E_p$ in Step 2.

Step 4: Compute inverse
$A^{-1} = E_p \cdots E_2 E_1$.

Algorithm 2:

Apply GE to $[A|I_n]$
$[A|I_n] \xrightarrow{op_1} \square \xrightarrow{op_2} \cdots \xrightarrow{op_p} [I_n|A^{-1}]$.

# XI. Linear Space

· Linear Space

Def. Suppose $V$ is a set associated with two operations:
(i) Addition "+": $u+v \in V$, $\forall u \in V, v \in V$.
(ii) Scalar multiplication: $\alpha u \in V$, $\forall \alpha \in R$, $u \in V$.
$V$ is called a linear space over $R$ if the 8 axioms hold ($u, v, w \in V$, $\alpha, \beta \in R$).
1. $u+v = v+u$
2. $u+(v+w) = (u+v)+w = u+v+w$
3. There exists a element $0$ s.t. $u+0 = u$.
4. $-u = (-1)\cdot u$, s.t. $u+(-u) = 0$
5. $\alpha(u+v) = \alpha u + \alpha v$
6. $(\alpha+\beta)u = \alpha u + \beta u$
7. $\alpha(\beta u) = (\alpha\beta)u$
8. $1u = u$.

· Euclidean Space

Euclidean space is a linear space, when equipped with addition and scalar-vector product.
e.g. $R^n$ is called n-dimensional Euclidean space.

· Polynomial Space

Set of polynomials with degree no more than $k$ is a linear space.

- Subspace

Def. Suppose $V$ is a linear space. We say $W$ is a subspace of $V$ if two conditions hold:
1. $W$ is a subset of $V$
2. $W$ is a linear space

In words: A subspace of $V$ is a subset that is itself a linear space.

Property: A subspace $W$ must contain the zero element: $0 \in W$

## XII. Linear Independence

- Linear Dependence and Independence

Def. Suppose $V$ is a linear space over $\mathbb{R}$. $u_1, u_2, \cdots, u_k \in V$.
We say $u_1, u_2, \cdots u_k$ are
1. linearly dependent if
   $\exists c_1, c_2, \cdots, c_k \in \mathbb{R}$ s.t. $(c_1, c_2, \cdots, c_k) \neq (0, 0, \cdots, 0)$ and $c_1u_1 + c_2u_2 + \cdots + c_ku_k = 0$.
2. linearly independent if
   $\forall c_1, c_2, \cdots, c_k \in \mathbb{R}$ s.t. $(c_1, c_2, \cdots, c_k) \neq (0, 0, \cdots, 0)$ and $c_1u_1 + c_2u_2 + \cdots + c_ku_k \neq 0$.

In short, $c_1u_1 + c_2u_2 + \cdots + c_ku_k = 0$ only happens when $c_1 = c_2 = \cdots = c_k = 0$

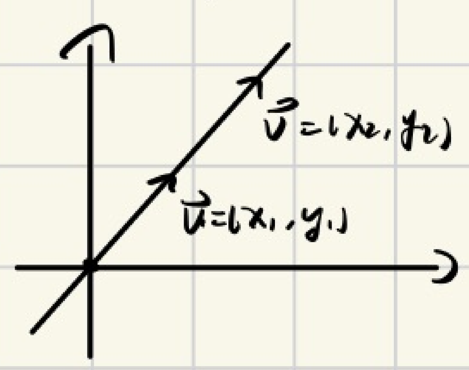
- Checking Linear Independence

If the space is $\mathbb{R}^n$, then just need to solve a linear system!

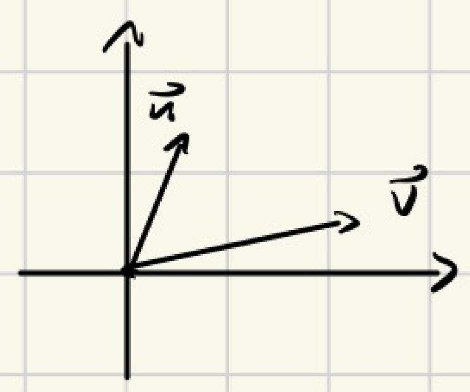
If the space is not $\mathbb{R}^n$, need extra tools. (not discuss here).

- Geometry

In the plane:



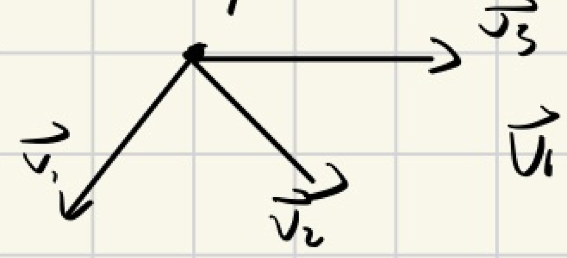

$\vec{u}$ & $\vec{v}$ linearly dependent




$\vec{u}$ & $\vec{v}$ linearly independent

In a 3D space:

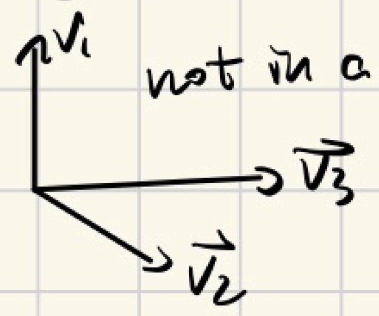
in a plane




$\vec{v_1} + \vec{v_3} - \vec{v_2} = 0$

linear dependent vectors

not in a plane




linear independent vectors

## XIII. Null Space

- Homogeneous Linear System

Def. A homogeneous linear system is
$$Ax = 0$$
where $A \in \mathbb{R}^{m \times n}$ are given and $x \in \mathbb{R}^{n \times 1}$ is the variable.

Theorem:

The solution set of a homogeneous linear system $Ax=0$ is a linear space (also a subspace of $R^n$).

Def. The solution set of $Ax=0$ is called the null space of $A$, denoted as $N(A)$.

## XIV. Span

### · Two Ways to Generate Subspaces

1. Solution set of linear equation + Taking intersection
   $\Rightarrow$ null space

2. Linear combination $\Rightarrow$ span

### · Span

Def. Suppose $V$ is a linear space, $u = \{u_1, u_2, \cdots, u_k\}$ is a subset of $V$.
The span of $\vec{u}$ is defined as

$\triangle$ $\text{span}(u) \triangleq \{a_1u_1 + a_2u_2 + \cdots + a_ku_k \mid a_1, a_2, \cdots, a_k \in R\}$.

In words: the span of elements of a linear space is the set of all linear combinations.

Fact: The span of any finite subset of $V$ is a subspace of $V$.

e.g. RGB as a spanning set.
The set $\{R, G, B\}$ spans the entire color space.

### · Span of Unit Vectors

Def. Consider two point $e_1 = (0,1)$, $e_2 = (1,0)$ on the plane.
$\text{span}\{(1,0),(0,1)\} = R^2$.
Consider $n$ vectors $e_1, e_2, \cdots, e_n$ are linearly independent.
Then $\text{span}\{e_1, e_2, \cdots, e_n\} = R^n$.

### · Spanning Set

Def. Suppose $V$ is a linear space, $u = \{u_1, u_2, \cdots, u_k\}$ is a subset of $V$.
If $\text{span}(u) = V$, then we say $u$ is a spanning set of $V$, or $u$ spans $V$.

e.g. $\{e_1, e_2, \cdots, e_n\}$ is a spanning set of $R^n$.

## XV. Solving $Ax = b$

### · Column Space

Def. Suppose $A = [a_1, a_2, \cdots, a_n] \in R^{m \times n}$ is a matrix.
The $\text{span}(\{a_1, a_2, \cdots, a_n\})$ is called the column space of $A$, denoted as $C(A)$.
In words: A's column space is the span of A's column vectors.

$\triangle$ Solvability: $Ax = b$ has a solution iff $b \in C(A)$.

- Solution Set of $Ax=b$

Def. Suppose $V$ is a linear space. For any element $v \in V$, and a subspace $U$, define
$v+U \triangleq \{v+u \mid u \in U\}$.

Proposition:
The solution set of a linear system $Ax=b$ is:
$\begin{cases} \text{either (i) an empty set} \\ \text{or (ii) } x_p + N(A) \text{, where } x_p \text{ is any solution of } Ax=b. \end{cases}$

# XVI. Basis

- Redundancy of Set

Theorem: Dependent Set is Redundant
Suppose $M = \{u_1, u_2, \cdots, u_k\}$ is linear dependent, then $\exists H \subsetneqq M$ s.t. $\text{span}(M) = \text{span}(H)$. (strictly included)
In words, it can be reduced without changing its span.

By contrast, if $M$ is independent, then there's no set $H \subsetneqq M$ s.t. $\text{span}(M) = \text{span}(H)$.
In words, removing any element changes the span.

- Basis

Def. Suppose $V$ is a linear space over $\mathbb{R}$, $u \triangleq \{u_1, u_2, \cdots, u_k\} \subseteq V$.
We say $u$ is a basis if:
$\begin{cases} 1.\ u \text{ is linearly independent.} \\ 2.\ \text{span}(u) = V. \end{cases}$
In words, a basis of $V$ is linearly independent & spanning set of $V$.
A basis of $V$ is a minimal spanning set of $V$ in the sense that we can't further shrink the set while being a spanning set of $V$.

- Dimension

Def. Dimension = Size of Bases.

Theorem: Bases have same size
If $\{u_1, u_2, \cdots, u_m\}$ and $\{v_1, v_2, \cdots, v_n\}$ are bases of a linear space $V$, then $m=n$.

Def. Suppose $V$ is a linear space
If $V$ has a basis $u$ with $n$ elements, then we say the dimension of $V$ is $n$.
Denoted as $\dim(V) = n$, or $V$ is $n$-dimensional.

- Quick way to check basis

Proposition:
Consider a linear space $V$ with $\dim(V) = n$.

$n$ vector in $\mathbb{R}^n$ $\begin{cases} \text{Independence} \\ \text{Span} \end{cases} \Rightarrow$ Basis

- Invertibility Conditions

Theorem: Equivalent Conditions for Invertibility
Let $A \in R^{n\times n}$. The following statements are equivalent.
1. $A$ is invertible.
2. The linear system $Ax=0$ has a unique solution $x=0$.
3. $A$ is a product of elementary matrices
4. $A$ has $n$ pivots
5. The columns of $A$ are linearly independent.
6. The columns of $A$ span $R^n$.
7. The columns of $A$ form a basis
8. $\dim(C(A)) = n$, $\text{rank}(A) = n$.
9. $Ax = b$ is solvable for any $b$.

## XVII. Rank

- Row Space

Def. Consider a matrix $A = \begin{bmatrix} a_{(1)}^T \\ a_{(2)}^T \\ \vdots \\ a_{(m)}^T \end{bmatrix} \in R^{m\times n}$

$\text{span}(\{a_{(1)}, a_{(2)}, \cdots, a_{(m)}\}) \subseteq R^n$ is called the row space of $A$, denoted as $Row(A)$

$A$'s row space is the span of $A$'s row vectors: $Row(A) = C(A^T)$.

- Rank

Def. Row-rank of $A$ is defined as $\dim(Row(A))$, denoted as $r_R(A)$.
Column-rank of $A$ is defined as $\dim(C(A))$, denoted as $r_C(A)$.

Theorem: Row-rank, column-rank and rank of a matrix are the same.
$r_R(A) = r_C(A) = \text{rank}(A)$.
$\Leftrightarrow \dim(Row(A)) = \dim(C(A))$.

Def. The rank of a matrix $A$ is $r_R(A)$, which is also $r_C(A)$, denoted as $\text{rank}(A)$.
For $A \in R^{m\times n}$, $\text{rank}(A) \le \min\{m, n\}$.

- Rank and Pivots

Theorem: Suppose $M$ is a RREF.
Then $r_R(M) = r_C(M)$ = Number of pivots = $\text{rank}(A)$.

- Full Rank Matrix

Def. Let $A \in R^{m\times n}$.
If $\text{rank}(A) = m$, then we say $A$ has full row rank.
If $\text{rank}(A) = n$, then we say $A$ has full column rank.
If $\text{rank}(A) = \min\{m, n\}$, then we say $A$ has full rank.

- Rank and Solutions

Def. Consider a matrix $A \in \mathbb{R}^{m\times n}$.

1. If $\text{rank}(A) = n$ (full column rank), then for any $b$, $Ax = b$ has at most one solution.
   (# of solutions must be 0 or 1, eliminate possibility of $\infty$).
   This means: No free columns in RREF => No free variables
2. If $\text{rank}(A) = m$ (full row rank), then $Ax = b$ has at least one solution.
   (# of solutions must be $\overset{n=m}{1}$ or $\overset{n>m}{\infty}$, eliminate possibility of 0).
   This means: No "zero rows" in RREF.

# XVIII. Orthogonality

- Orthogonality

Def. Two vectors $u, v \in \mathbb{R}^n$ are orthogonal if $u^T \cdot v = 0$. Denote $u \perp v$.

And $\cos\theta = \frac{u^T \cdot v}{\|u\| \cdot \|v\|} = 0$

We say a vector $u \in \mathbb{R}^n$ is orthogonal to a subspace $V \subseteq \mathbb{R}^n$ if $u \perp v$, $\forall v \in V$. Denote $u \perp V$.

We say two subspaces $U, V \subseteq \mathbb{R}^n$ are orthogonal if $u \perp v$, $\forall u \in U, v \in V$. Denote $U \perp V$.

- Orthogonal Complement

Def. For any subspace $V \subseteq \mathbb{R}^n$, the set of vectors that are orthogonal to $V$
$\{u \in \mathbb{R} \mid u \perp v, \forall v \in V\}$.
is called the orthogonal complement of $V$, denoted as $U = V^\perp$

Remark: A subspace $V$ has a unique orthogonal complement.

Theorem: Dimension of $V$ prep.

Suppose $S$ is a subspace of $\mathbb{R}^n$. Then the following hold:

i) $S^\perp$ is a subspace
ii) $\dim(S) + \dim(S^\perp) = n$
iii) If $\{u_1, u_2, \dots, u_r\}$ is a basis of $S$, and $\{u_{r+1}, \dots, u_n\}$ is a basis of $S^T$, then $\{u_1, \dots, u_r, u_{r+1}, \dots, u_n\}$ is a basis of $\mathbb{R}^n$.

# XIX. Four Fundamental Subspaces

- Left Null Space

Def. The left null space of a matrix $A$ is defined as $N(A^T)$.

- Fundamental Theorem

Theorem: Suppose $A \in \mathbb{R}^{m\times n}$. Then:

1. $N(A) = \text{Row}(A)^\perp$, $\dim(N(A)) = n - \text{rank}(A)$.
2. $N(A^T) = C(A)^\perp$, $\dim(N(A^T)) = m - \text{rank}(A)$.

# XX. Least Square

## Linear Regression

Minimization Problem

Method: Make error as small as possible

Find $x$ s.t. $\|Ax-b\|$ is the smallest among all.

More precisely, find $x^*$ s.t. $\|Ax^*-b\| \leq \|Ax-b\|$, $\forall x$.

## Least Squares

Def. Given $A \in R^{m\times n}$, $b \in R^{m\times 1}$, the least square problem is

$$\min_{x\in R^{n\times 1}} \|Ax-b\|$$

$y$ is a solution of the problem if

$\|Ay-b\| \leq \|Ax-b\|$, $\forall x$.

And $y$ is called the regression vector

The method to find $y$ called linear regression.

Suppose $a_{(1)}^T, \cdots, a_{(n)}^T$ are rows of $A$.

$\|Ax-b\|^2 = (a_{(1)}^T x - b_1)^2 + \cdots + (a_{(n)}^T x - b_n)^2$

## Solving LS is Finding Projection

Lemma: Orthogonal Projection

Suppose $S$ is a subspace of $R^m$. Suppose $p \in S$, then

$\|b-p\| \leq \|b-z\|$, $\forall z \in S \iff b-p \perp S$

Def. Suppose $S$ is a subspace of $R^m$.

Suppose $p \in S$ and $b-p \perp S$, then we say $p$ is the projection of $b$ onto $S$.

## Finding Projection is Solving Equation

Lemma:

Suppose $S = C(A)$ is the column space of matrix $A$.

Then: $b - Ay \perp C(A) \iff A^TAy = A^Tb$.

Theorem: LS solution and normal equation

Given $A \in R^{m\times n}$, $b \in R^{m\times 1}$. The following statements are equivalent:

1. $y$ is a solution of the problem $\min_{x\in R^{n\times 1}} \|Ax-b\|$.
2. $A^TAy = A^Tb$

## Analyzing Normal Equation

Proposition (Existence):

The linear system $A^TAy = A^Tb$ has at least one solution.

Corollary (Characterization)

Suppose $A$ has linearly independent columns (i.e. has full column rank),

then $A^TA$ is invertible solution of the least square problem $\min_{x\in R^{n\times 1}} \|Ax-b\|$ is $y = (A^TA)^{-1}A^Tb$.

# XXI. Orthonormal Basis

- **Orthonormal Set of Vectors**

Def. Let $\{v_1, v_2, \cdots, v_k\}$ be a set of nonzero vectors in $R^n$. If $\langle v_i, v_j \rangle = 0$ for any $i \neq j$, then this set is called an orthogonal set.

An orthonormal set $\{v_1, v_2, \cdots, v_k\}$ is an orthogonal set of unit-norm vectors. i.e. $\|v_i\| = 1$ for all $i = 1, \cdots, n$.

Proposition: If $S = \{v_1, v_2, \cdots, v_k\}$ is an orthogonal set, then $v_1, \cdots, v_k$ are linearly independent.

- **Orthogonormal Basis**

Def. A set of vectors $S = \{u_1, \cdots, u_n\}$ is called an orthogonormal basis of a linear space $V$ if
1. $S$ is an orthonormal set.
2. Vector in $S$ form a basis of $V$.

- **Orthogonal Matrix**

Def. An orthogonal matrix $Q \in R^{n \times n}$ is a real square matrix whose columns form an orthonormal basis in $R^n$.

A matrix $Q \in R^{n \times n}$ is an orthogonal matrix iff $Q^T Q = I_n$.

Corollary:
If $Q \in R^{n \times n}$ is orthogonal then $Q^{-1} = Q^T$.

Properties:
1. $\langle Qx, Qy \rangle = \langle x, y \rangle$
2. $\|Qx\| = \|x\|$.

- **Representation Elements Using Basis**

Proposition (Unique Representation via Basis)
Let $u = \{u_1, \cdots, u_n\}$ be a basis of a linear space $V$.
Any element $v \in V$ can be uniquely represented as a linear combination of $u_1, \cdots, u_n$.
Representation: $v = a_1 u_1 + \cdots + a_n u_n$.

Theorem: Representation via Orthonormal Basis
Let $u = \{u_1, \cdots, u_n\}$ be an orthonormal basis of $R^n$.
Any vector $v \in R^n$ can be represented as a linear combination

$$v = \sum_{i=1}^{n} u_i u_i^T v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i .$$

e.g. $\left\{ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis of $R^3$.

For any $v = [x, y, z]^T$, one has

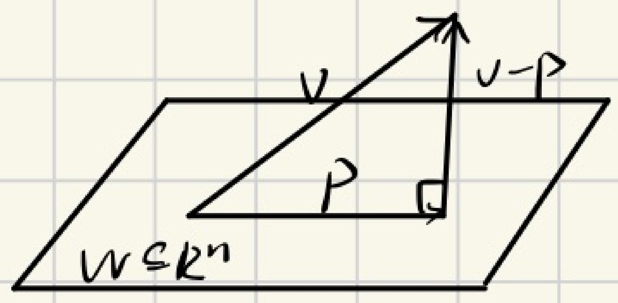
$$v = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \langle v, u_3 \rangle u_3 .$$

$$= \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

$$= [x, y, z]^T .$$

- Projection

Def. Suppose $S$ is a subspace of $R^m$.
Suppose $p \in S$ and $b-p \perp S$, then we say $p$ is the projection of $b$ onto $S$.




Proposition:
For any subspace $W \subset R^n$, any vector $v \in R^n$, there is a unique vector $p \in W$ s.t. $(v-p) \perp W$.

Theorem: Projection when Ortho-Basis Exists
Let $W$ be a subspace of $R^n$ and suppose $S = \{w_1, \cdots, w_k\}$ is an orthonormal basis of $W$.
The projection of $v$ onto $W$ can be represented as
$$p = \sum_{i=1}^{k} \langle w_i, v \rangle w_i .$$

- Gram-Schmidt Process

Def. Suppose $U = \{u_1, \cdots, u_k\}$ is a set of linearly independent elements.
Input $U = \{u_1, \cdots, u_k\}$.
For $i = 1, \cdots, k$
If $i = 1$:
$p_1 = 0$
Else:
$$p_{i-1} = \sum_{j=1}^{i-1} \langle u_i, v_j \rangle = proj_{span(u_1, \cdots, u_{i-1})}(u_i) .$$
Set $v_i = \dfrac{u_i - p_{i-1}}{\| u_i - p_{i-1} \|} \in span(u_1, \cdots, u_{i-1})^{\perp}$.

Return $V = \{v_1, \cdots, v_k\}$.

Proposition:
The set $V = \{v_1, \cdots, v_k\}$ returned by the Gram-Schmidt process is an orthonormal basis of span $(U)$.

## XXII. Determinant