MAT

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MAT 1002 Calculus II
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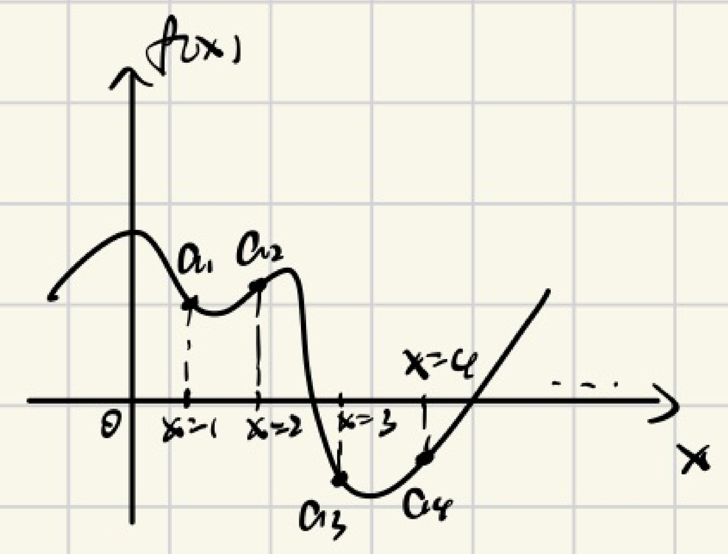
I. Infinite Sequences and Series.

10.1 Sequences.



- **Representing Sequences**

Def. A set of sample of a function $f(x)$, which $a_n = f(x=n)$.
A list of numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$
The integer n is the index of a_n
An infinite sequence of numbers is a function whose domain is N^*.
e.g. $a_n = \sqrt{n}$.

$$\Rightarrow \{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \cdots, \sqrt{n}, \cdots\}.$$
$$= \{\sqrt{n}\}_{n=1}^{\infty}.$$

P.S. we are only interested in infinite sequences.

- **Convergence and Divergence**

Def. The sequence $\{a_n\}$ converges to the number L if for every positive number ϵ there corresponds an integer N such that for all n,
$$n > N \Rightarrow |a_n - L| < \epsilon.$$
If no such number L exists, we say that $\{a_n\}$ diverges.
If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$, and call L the limit of $\{a_n\}$.

Def. The sequence $\{a_n\}$ diverges to infinity if for every number M there is an integer N s.t. for all $n > N$, $a_n > M$, we write:
$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty.$$
(negative infinity is similar).

- **Calculating Limits of Sequences.**

Theorem 1: Let $\{a_n\}$, $\{b_n\}$ be sequences of R, and let $A, B \in R$.
If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, we have:

① $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$ — Sum & Difference Rule

② $\lim_{n\to\infty} (k\cdot a_n) = k\cdot A$ $(\forall k \in R)$. — Constant Multiple Rule

③ $\lim_{n\to\infty} (a_n \cdot b_n) = A\cdot B$ — Product Rule

④ $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$. — Quotient Rule.

Theorem 2: The Sandwich Theorem for Sequences.
Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of R. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. We have $\lim_{n\to\infty} b_n = L$.

e.g. Since $\frac{1}{n} \to 0$, we have
1° $\frac{\cos n}{n} \to 0$ because $-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$.
2° $\frac{1}{2^n} \to 0$ because $0 \le \frac{1}{2^n} \le \frac{1}{n}$.
3° $(-1)^n \frac{1}{n} \to 0$ because $-\frac{1}{n} \le (-1)^n \frac{1}{n} \le \frac{1}{n}$.

Theorem 3: The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of $\mathbb{R}$. If $a_n \to L$ and f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

▷ $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$.

Theorem 4: Suppose that $f(x)$ is a function defined for all $x \geq n_0$, and that $\{a_n\}$ is a sequence of $\mathbb{R}$ s.t. $a_n = f(n)$ for $n \geq n_0$, then:

▷ $\lim_{n\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$. Treat n as x.

e.g. Show that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$. L'Hôpital's Rule

A: By theorem 4. We have $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$.

So that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

- **Commonly Occurring Limits**

① $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

② $\lim_{n\to\infty} \sqrt[n]{n} = 1$

③ $\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \quad (x>0)$.

④ $\lim_{n\to\infty} x^n = 0 \quad (|x|<1)$.

⑤ $\lim_{n\to\infty} (1+\frac{x}{n})^n = e^x \quad (\forall x \in \mathbb{R})$.

⑥ $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\forall x \in \mathbb{R})$.

- **Bounded Monotonic Sequences**

Def. A sequence $\{a_n\}$ is bounded from above is $\exists M$ s.t. $a_n \leq M$ for all n. We called M is an upper bound for $\{a_n\}$. And M is the least upper bound if it's the smallest.
(lower bound & greatest lower bound is similar).

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ ~~~~ unded, unless it's unbounded.

Def. A sequence $\{a_n\}$ is nondecreasing if $a_n \leq a_{n+1}$ for all n, that is, $a_1 \leq a_2 \leq a_3 \leq \cdots$
(nonincreasing is similar)
The sequence $\{a_n\}$ is monotonic if it is either nondecreasing or nonincreasing.

Theorem 6: The Monotonic Sequence Theorem.

▷ If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

- **Recursive Definitions**

Sequences are often defined recursively by giving

① The value(s) of the initial term or terms

② A rule, called recursion formula, for calculating the later terms.

e.g. Fibonacci numbers.

10.2 Infinite Series

Concepts

Def. Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an infinite series. a_n is the nth term of the series.

And we have $S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_n$. which is the nth partial sum.

If the sequence of partial sums converges to a limit L, we say that the series converges. we write:

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

otherwise, the series diverges.

Geometric Series

Def. Geometric series are series of the form:

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n \quad (a \neq 0).$$

▷ And $S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \quad (r \neq 1)$.

If $|r| < 1$, the geometric series converges to $\frac{a}{1-r}$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

The nth-Term Test

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

▷ Test: $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from 0.

e.g. $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$.

$\sum_{n=1}^{\infty} (-1)^n$ diverges because $\lim_{n\to\infty} (-1)^n$ DNE.

Combining Series

Theorem:

If both $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

① $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$ — Sum and Difference Rule

② $\sum k a_n = k \cdot \sum a_n = kA \quad (k \in R)$. — Constant Multiple Rule

If $\sum a_n$ diverges and $\sum b_n$ converges,

① $k \cdot \sum a_n$ diverges $(k \neq 0)$.

② $\sum (a_n \pm b_n)$ diverges.

If both $\sum a_n$ and $\sum b_n$ diverges, $\sum (a_n \pm b_n)$ is not for sure.

e.g. $\sum a_n = 1 + 1 + 1 + \cdots$, $\sum b_n = (-1) + (-1) + (-1) + \cdots$, $\sum (a_n + b_n) = 0$, which converges.

but $\sum (a_n + a_n)$ diverges.

10.3 The Integral Test

· Nondecreasing Partial Sums

Theorem: A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if and only if its partial sums are bounded from above

$a_n \geq 0$ for all n. $\Rightarrow S_1 \leq S_2 \leq S_3 \leq \cdots \leq S_n \leq S_{n+1} \leq \cdots$

(Corollary of the Monotonic Sequence Theorem).

· The Integral Test

Theorem: Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of $x \geq N$.

Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.

e.g. Show that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$ converges if $p>1$, diverges if $p \leq 1$.

1° If $p>1$. then $f(x) = \frac{1}{x^p}$ is a positive decreasing function of x.

Since $\int_1^{\infty} \frac{1}{x^p}dx = \lim_{b\to\infty} \int_1^b x^{-p}dx = \frac{1}{p-1}$, which converges.

we have the series also converges.

2° If $p \leq 0$. since $\lim_{n\to\infty} a_n \neq 0$, the series diverges.

3° If $0<p<1$. then $1-p>0$.

Since $\int_1^{\infty} \frac{1}{x^p}dx = \frac{1}{1-p} \lim_{b\to\infty}(b^{1-p}-1) = \infty$, which diverges.

we have the series diverges.

4° If $p=1$. $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots \geq 1 + \frac{1}{2} + (\frac{1}{4}+\frac{1}{4}) + (\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}) + \cdots$

which diverges. (we called it the harmonic series).

· Error Estimation

Suppose that a series $\sum a_n$ is shown to be convergent, and we want to estimate the size of the remainder R_n measuring the difference between S and S_n. that is:

$R_n = S - S_n = a_{n+1} + a_{n+2} + \cdots$

According to the Integral Test, $a_k = f(k)$. where f is a continuous positive decreasing function for all $x \geq n$, and that $\sum a_n$ converges to S, we have:

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx.$$

If we add S_n to each side of the inequality, we get

$$S_n + \int_{n+1}^{\infty} f(x)dx \leq S \leq S_n + \int_n^{\infty} f(x)dx$$ $\Rightarrow$ Estimate S!

e.g. Estimate $\sum \frac{1}{n^2}$ and $n=10$.

We have that:

$$\int_n^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_n^b = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{n}\right) = \frac{1}{n}.$$

Thus $S_{10} + \frac{1}{11} \leq S \leq S_{10} + \frac{1}{10}$.

$1.64068 \leq S \leq 1.64977$.

If we approximate S by the midpoint. we find that $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.6452$.

10.4 Comparison Test

· **Definition**

Theorem: The Comparison Test

Let $\sum a_n$, $\sum c_n$ and $\sum d_n$ be series with nonnegative terms.
Suppose that for some N: $d_n \le a_n \le c_n$ for all $n > N$.
① If $\sum c_n$ converges, then $\sum c_n$ also converges.
② If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

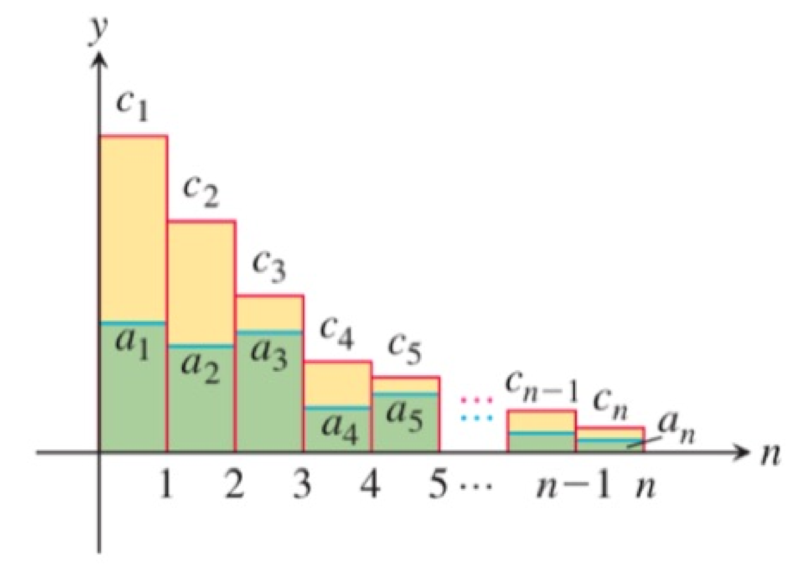


FIGURE 10.12 If the total area $\sum c_n$ of the taller c_n rectangles is finite, then so is the total area $\sum a_n$ of the shorter a_n rectangles.

e.g. The series $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges because

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}.$$

And $\sum \frac{1}{n}$ diverges due to harmonic series.

· **The Limit Comparison Test**

Theorem: Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$.
① If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

② If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

③ If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

e.g. Is $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ convergent or divergent?

Let $a_n = \frac{1}{2^n - 1}$. For large n, we expect a_n to behave like $\frac{1}{2^n}$. So we let $b_n = \frac{1}{2^n}$.
Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = 1$.

$\sum a_n$ converges by Part 1 of the Limit Comparison Test.

10.5 Absolute Convergence

· **Definition**

A series $\sum a_n$ converges absolutely (is absolutely convergent) if the the corresponding series of absolute values, $\sum |a_n|$, converges.

Theorem: The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

e.g. For $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, which contains both positive and negative terms,

the corresponding series of absolute value is:

$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots$ which converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

because $|\sin n| \le 1$ for every n.

The original series converges absolutely, therefore it converges.

- The Ratio Test

Theorem: The Ratio Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$$

Then ① the series converges absolutely is $\rho < 1$.
② the series diverges if $\rho > 1$ or ρ is infinite
③ the test is inconclusive if $\rho = 1$.

e.g. Investigate the convergence of $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\cdot n!}$.

If $a_n = \frac{(2n)!}{n!\cdot n!}$, $a_{n+1} = \frac{(2n+2)!}{(n+1)!\cdot(n+1)!}$ and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n!\,n!\,(2n+2)(2n+1)(2n)!}{(n+1)!\,(n+1)!\,(2n)!} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4.$$

The series diverges since $\rho = 4 > 1$ by the Ratio Test.

- The Root Test

Theorem: The Root Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \rho$$

Then ① the series converges absolutely if $\rho < 1$.
② the series diverges if $\rho > 1$ or ρ is infinite.
③ the test is inclusive if $\rho = 1$.

e.g. Investigate the convergence of $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$.

$\lim_{n\to\infty}\sqrt[n]{n} = 1$.

Since $\lim_{n\to\infty}\sqrt[n]{\left|\frac{n^2}{2^n}\right|} = \lim_{n\to\infty}\frac{\sqrt[n]{n^2}}{2} = \lim_{n\to\infty}\frac{(\sqrt[n]{n})^2}{2} = \frac{1^2}{2} = \frac{1}{2} < 1$.

The series converges due to $\rho = \frac{1}{2}$ by the Root Test.

10.6 Alternating Series and Conditional Convergence

- Alternating Series

Def. A series which the terms are alternately positive and negative.

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$ (alternating harmonic series).

$1 - 2 + 3 - 4 + \cdots + (-1)^{n+1} + \cdots$

Theorem: The Alternating Series Test.

The series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$ converges if all of the following satisfied:

1. The u_n's are all positive
2. The u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n > N$, for some N.
3. $u_n \to 0$.

Theorem: The Alternating Series Estimate Theorem

If $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ converges, then for all $n \geq N$,

$S_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$.

approximate the sum S of the series with an error e that $|e| < u_{n+1}$.

- Conditional Convergence

Def. A convergent series that is not absolutely convergent is conditionally convergent.

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$, $p>0$. converges.

① If $p>1$, the series converges absolutely as an ordinary p-series.
② If $0<p\le 1$, the series converges conditionally by the alternating series test.

- Rearranging Series

Theorem: The Rearrangement Theorem for Absolutely Convergent Series
If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n \ldots$ is an arrangement of $\{a\}$, then

$\sum b_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

Summary of Tests

We have developed a variety of tests to determine convergence or divergence for an infinite series of constants. There are other tests we have not presented which are sometimes given in more advanced courses. Here is a summary of the tests we have considered.

1. **The *n*th-Term Test:** If it is not true that $a_n \to 0$, then the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. ***p*-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

10.7 Power Series

- Power Series and Convergence

Def. A power series about $x=0$ is a series of the form

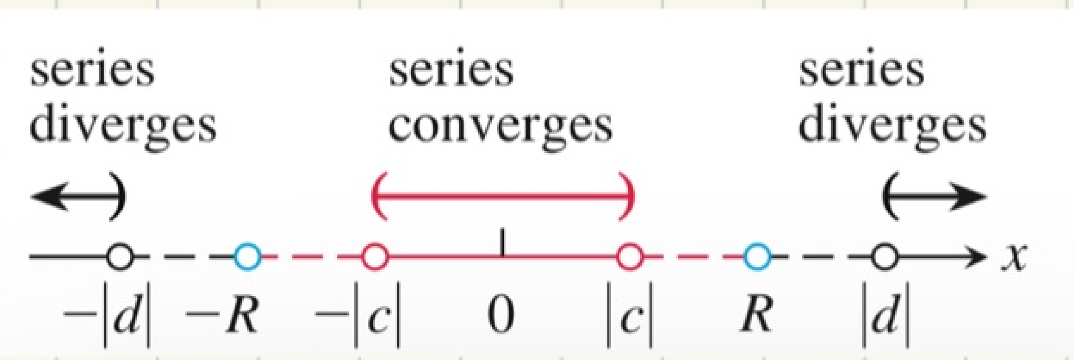
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

A power series about $x=a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the center a and the coefficients $c_0, c_1, c_2, \ldots c_n, \ldots$ are constants.

Theorem: The Convergence Theorem for Power Series
If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x=c\neq 0$, then it converges absolutely for all x with $|x|<|c|$. If the series diverges at $x=d$, then it diverges for all x ($|x|>|d|$).



- ## The Radius of Convergence of a Power Series

Theorem: The convergence of the series $\sum c_n(x-a)^n$ is describe by one of the following cases:

1. There is a positive number R s.t. the series diverges for x with $|x-a|>R$ but converges absolutely for x with $|x-a|<R$. The series may or may not converge at either of the endpoints $x=a-R$ and $x=a+R$.
2. The series converges absolutely for every x ($R=\infty$).
3. The series converges at $x=a$ and diverges elsewhere ($R=0$).

R is called the radius of convergence of the power series, and the interval of radius R centered at $x=a$ is called the interval of convergence.

Q: How to test a power series for convergence?

A:
1. Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely. Ordinarily, this is an open interval $|x-a|<R$
2. If the interval is finite, test for convergence or divergence at each endpoint.
3. If the interval is $|x-a|<R$, the series diverges for $|x-a|>R$.

- ## Operations on Power Series

Theorem: The Series Multiplication Theorem for Power Series.

If $A(x)=\sum_{n=0}^{\infty} a_n x^n$ and $B(x)=\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x|<R$, and $c_n=\sum_{k=0}^{n} a_k b_{n-k}$,

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x|<R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)=\sum_{n=0}^{\infty} c_n x^n.$$

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)|<R$.

Theorem: The Term-by-Term Differentiation Theorem

If $\sum c_n(x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n \quad \text{on the interval} \quad a-R<x<a+R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$\begin{cases} f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}. \\ f''(x)=\sum_{n=1}^{\infty} n(n-1)c_n(x-a)^{n-2}, \end{cases}$$

and so on. Each of these derived series converges at every point of the interval $(a-R, a+R)$.

Theorem: The Term-by-Term Integration Theorem

Suppose that $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $a-R<x<a+R$ ($R>0$). Then

$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ converges for $a-R<x<a+R$ and

$$\int f(x)dx=\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}+C \quad \text{for } a-R<x<a+R.$$

10.8 Taylor and Maclaurin Series

· Series Representations

$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$ with a positive radius of convergence.

By repeated term-by-term differentiation with interval I, we obtain:

$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots,$

$f''(x) = 1\cdot 2a_2 + 2\cdot 3a_3(x-a) + 3\cdot 4a_4(x-a)^2 + \cdots,$

$\vdots$

With the n^{th} derivative, for all n, being

$f^{(n)}(x) = n!a_n$ + a sum of terms with $(x-a)$ as a factor.

Since these eq.s all hold at $x=a$, we have:

$f'(a) = a_1$, $f''(a) = 1\cdot 2a_2$ ... In general, $f^{(n)}(a) = n!a_n \Rightarrow a_n = \frac{f^{(n)}(a)}{n!}$.

Thus, if f has a series representation, the the series is:

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$

· Taylor and Maclaurin Series

Def. Let f be a function with derivatives of all orders thoughout some interval containing a as an interior point. Then the Taylor series generated by f at $x=a$ is:

$\triangle$ $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$

The Maclaurin Series of f is the Taylor series generated by f at $x=0$, or

$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$

· Taylor Polynomials

Def. Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x=a$ is the polynomial

$\triangle$ $P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$.

e.g. Let $f(x) = e^x$, then

Taylor series generated by f at $x=0$ is

$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$

$= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots$

$= \sum_{k=0}^{\infty} \frac{x^k}{n!}$

The Taylor polynomial of order n at $x=0$ is

$P_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$.

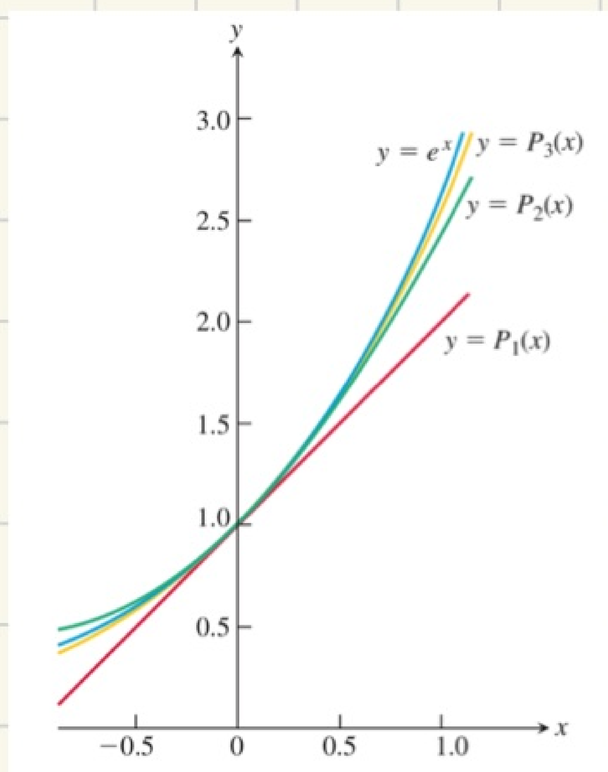


FIGURE 10.17 The graph of $f(x) = e^x$ and its Taylor polynomials

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + \frac{x^2}{2!}$

$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$\vdots$

It's getting more accurate.

e.g. Let $f(x) = \cos x$, then

The Taylor series generated by f at 0 is:

$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$

$= 1 + 0\cdot x - \frac{x^2}{2!} + 0\cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!}$

$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$.

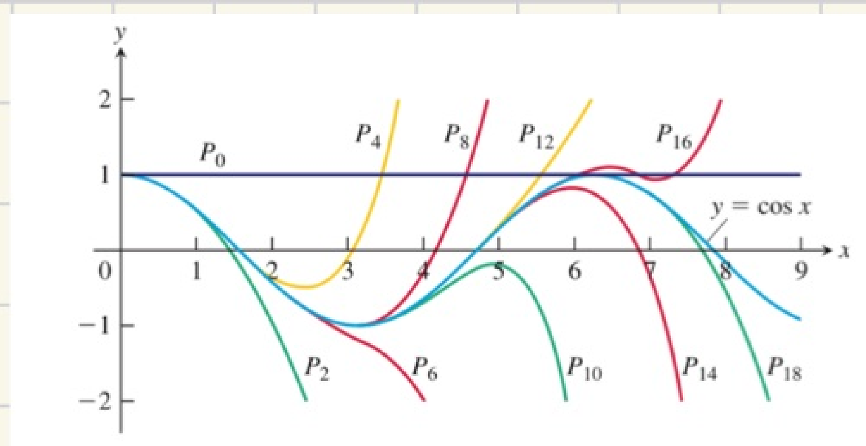


FIGURE 10.18 The polynomials

$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$

10.9 Convergence of Taylor Series

· Motivation

Theorem: Taylor's Theorem

If f and its first n derivatives $f', f'', \dots, f^{(n)}$ are continuous on the close interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Taylor's Theorem is a generalization of the Mean Value Theorem.

· Taylor's Formula

Def. If f has derivatives of all orders in an interval I containing a, then for each positive integer n and for each x in I.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x).$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

10.10 The Binomial Series and Applications of Taylor Series

· The Binomial Series for Powers and Roots

Def. For $-1<x<1$, $(1+x)^m = 1 + \sum_{k=1}^{\infty}\binom{m}{k}x^k$, is called the binomial series.

where we define $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2!}$, and $\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$ for $k \geq 3$.

The binomial series converges absolutely for $|x|<1$.

· Evaluating Nonelementary Integrals

Sometimes we can use a familiar Taylor series to find the sum of a given power series in terms of a known function. And it can be used to express unelementary integrals.

e.g. $\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \dots$

$$\int \sin x^2 dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \dots$$

It can be used for estimating.

e.g. For $\tan^{-1}x$, we have $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$, integrating to get:

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

We put $x=1$ into it and we get Leibniz's formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots + \frac{(-1)^n}{2n+1} + \dots$$

By multipying by 4, we get π.

· Evaluating Indeterminate Forms

We can sometimes evaluate indeterminate forms by express the functions as Taylor series.
e.g. Evaluate $\lim_{x\to 1} \frac{\ln x}{x-1}$.

We have $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots$.
So $\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} (1 - \frac{1}{2}(x-1) + \cdots) = 1$.

· Euler's Identity

Def. For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$. ☆

It enables us to define e^{a+bi} to be $e^a \cdot e^{bi}$ for any complex number $a+bi$.
One consequence of the identity is $e^{i\pi} + 1 = 0$

· Taylor Series

TABLE 10.1 Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

11. Parametric Equations and Polar Coordinates

11.1 Parametrizations of Plane Curves

· Parametric Equations

Def. If x and y are given as functions
$$x=f(t),\quad y=g(t)$$
over an interval I of t-values, then the set of points $(x,y)=(f(t),g(t))$ defined by these equations is a parametric curve. The equations are parametric equations for the curve.

The variable t is a parameter for the curve, and its domain I is the parameter interval. If I is a closed interval, $a\le t\le b$, the point $(f(a),g(a))$ is the initial point of the curve and $(f(b),g(b))$ is the terminal point.

e.g. $x=t^2$, $y=t+1$, $-\infty<t<\infty$.

TABLE 11.1 Values of $x=t^2$ and $y=t+1$ for selected values of t.

t	x	y
−3	9	−2
−2	4	−1
−1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

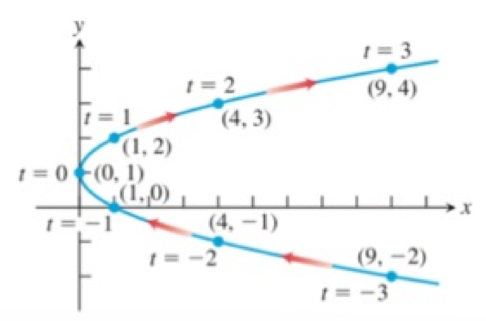


FIGURE 11.2 The curve given by the parametric equations $x=t^2$ and $y=t+1$ (Example 1).

· Cycloids

Motivation: A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference.
The path is called a cycloid.
We can get:
$$x=a(t-\sin t),\quad y=a(1-\cos t).$$

· Brachistochrones and Tautochrones

?

11.2 Calculus with Parametric Curves

· Tangents and Areas

A parametrized curve $x=f(t)$ and $y=g(t)$ is differentiable at t if f and g are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives are related by the Chain Rule:
$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}.$$

If $\frac{dx}{dt}\neq 0$, we can have:
$$\triangle\quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
Which is the Parametric Formula for $\frac{dy}{dx}$.

If parametric equations define y as twice-differentiable function of x, we can get
$$\frac{d^2y}{dx^2}=\frac{\frac{dy'}{dt}}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

- **Length of a Parametrically Defined Curve**

Def. If a curve C is defined parametrically by $x=f(t)$ and $y=g(t)$, $a\le t\le b$, where f' and g' are continuous and not simultaneously zero on $[a,b]$, and C is traversed exactly once as t increases from $t=a$ to $t=b$, then the length of C is the definite integral

$$L=\int_a^b\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

e.g. Find the length of the circle of radius r defined parametrically by
$x=r\cos t$ and $y=r\sin t$, $0\le t\le 2\pi$.
As t varies from 0 to 2π, the circle is traversed exactly once, we have

$$L=\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^{2\pi}\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt=\int_0^{2\pi}r\,dt=r\cdot[t]_0^{2\pi}=2\pi r.$$

- **Areas of Surfaces of Revolution for Parametrized Curves**

Def. If a smooth curve $x=f(t)$, $y=g(t)$, $a\le t\le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows:

1. Revolution about the x-axis $(y\ge 0)$:

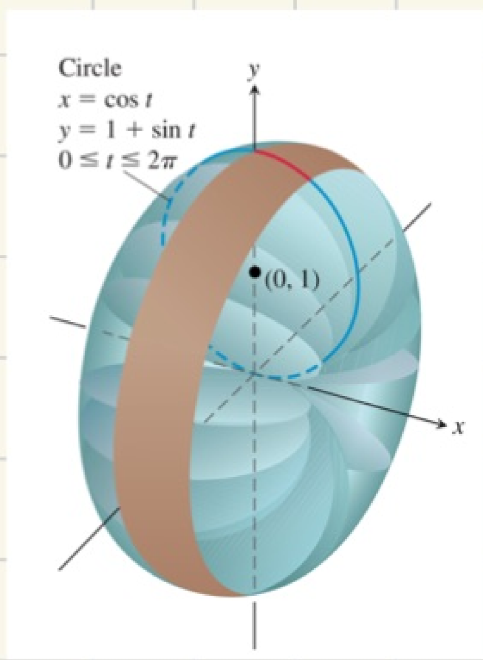
$$S=\int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

2. Revolution about the y-axis $(x\ge 0)$:

$$S=\int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

e.g. Find the area of the surface swept out by revolving the circle about the x-axis.
$x=\cos t$, $y=1+\sin t$, $0\le t\le 2\pi$.

We have
$$S=\int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$=\int_0^{2\pi}2\pi(1+\sin t)\sqrt{(-\sin t)^2+(\cos t)^2}\,dt$$
$$=2\pi\left[t-\cos t\right]_0^{2\pi}=4\pi^2.$$

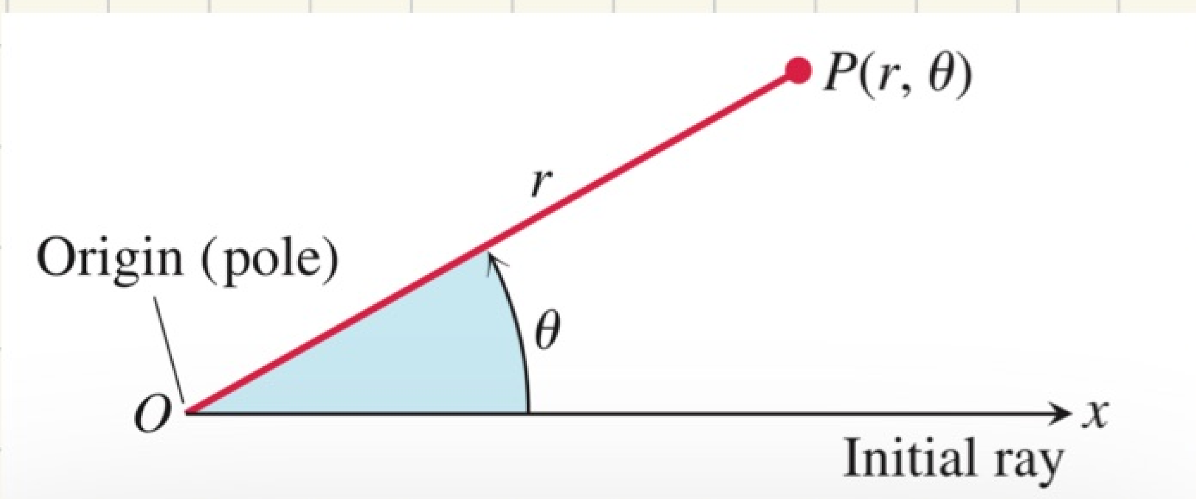


11.3 Polar Coordinates

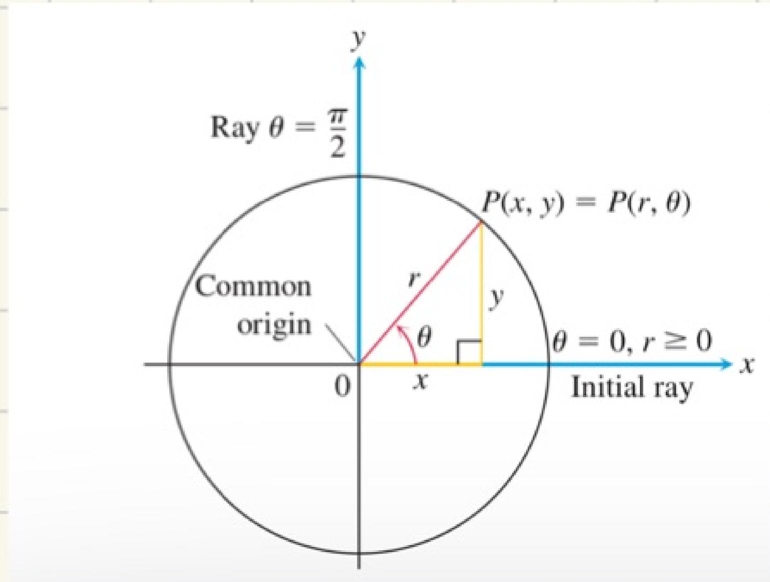
- **Definition of Polar Coordinates**

We first fix an origin O ("pole"), and an initial ray from O. Usually the positive x-axis is chosen as the initial ray. Then each point P can be located by assigning to it a polar coordinate pair (r,θ), in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP. We label the point P as

$P(r,\theta)$. r for distance, θ for angle.



Relating Polar and Cartesian Coordinates



Equations of x, y, r, θ.

$x = r\cos\theta \qquad r^2 = x^2 + y^2$

$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$

11.4 Graphing Polar Coordinate Equations

Symmetry

Symmetry Test for Polar Graphs in the Cartesian xy-plane:

1. x-axis: $(r, \theta) \to (r, -\theta)$ or $(-r, \pi - \theta)$.
2. y-axis: $(r, \theta) \to (r, \pi - \theta)$ or $(-r, -\theta)$.
3. origin: $(r, \theta) \to (-r, \theta)$ or $(r, \theta + \pi)$.

Slope

Slope of the Curve $r = f(\theta)$ in the Cartesian xy-plane.

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \qquad \left(\frac{dx}{d\theta} \neq 0 \text{ at } (r, \theta)\right).$$

Graphing Polar Curves Parametrically

We can convert polar function $r = f(\theta)$ into parametric form using the equations

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$.

Then we can draw a parametrized curve in the Cartesian xy-plane.

11.5 Areas and Lengths in Polar Coordinates

Area in the Plane

Def. Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$.

$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta.$$

This is the integral of the area differential

$$dA = \frac{1}{2} r^2 d\theta = \frac{1}{2} (f(\theta))^2 d\theta.$$

Def. Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$.

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 \, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 \, d\theta = \int_\alpha^\beta \frac{1}{2} (r_2^2 - r_1^2) \, d\theta.$$

Length of a Polar Curve

Def. If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is:

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

Ⅲ. Vectors and the Geometry Space

12.1 Three-Dimensional Coordinate Systems

· Distance and Spheres in Space

Distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.

$$|P_1P_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0).

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = a^2.$$

12.2 Vectors

All learnt in High School

12.3 Dot Products

All learnt in High School

The vector projection of $\vec{u}$ onto $\vec{v}$ is

Δ $\mathrm{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\cdot\vec{v}$.

The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is

$$|\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \vec{u}\cdot\frac{\vec{v}}{|\vec{v}|}.$$

12.4 The Cross Product

· The Cross Product of Two Vectors in Space

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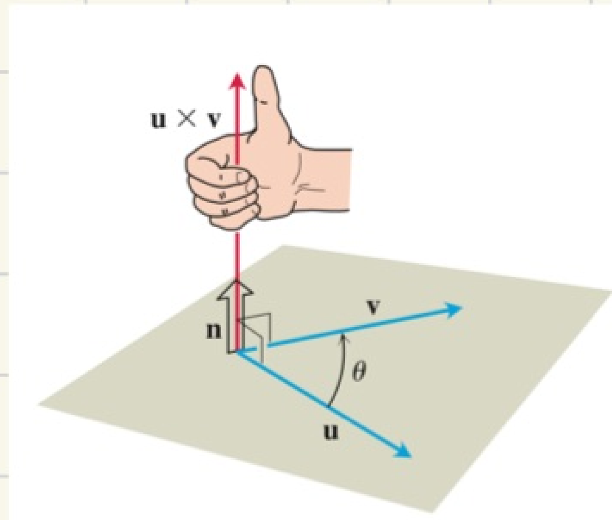


FIGURE 12.27 The construction of u × v.

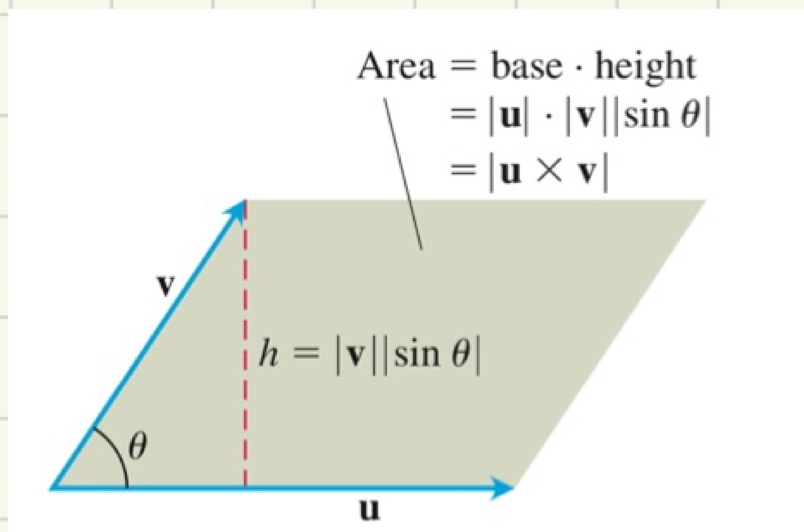
Def. The cross product $\vec{u}\times\vec{v}$ is the vecter

Δ $\vec{u}\times\vec{v} = (|\vec{u}|\cdot|\vec{v}|\cdot\sin\theta)\,\vec{n}$

Where $\vec{n}$ is a unit vector perpendicular to the plane of $\vec{u}$ and $\vec{v}$.

We have nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u}\times\vec{v} = 0$.

· The Area of a Parallelogram



Def. The area of a parallelogram determined by $\vec{u}$ and $\vec{v}$ is

$$|\vec{u}\times\vec{v}| = |\vec{u}|\cdot|\vec{v}|\cdot|\sin\theta|\cdot|\vec{n}| = |\vec{u}|\cdot|\vec{v}|\cdot\sin\theta.$$

· Determined Formula for $\vec{u}\times\vec{v}$

Def. Calculating the cross product as a determined:

If $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, then

$$\Delta\quad \vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}.$$

· Triple Scalar or Box Product

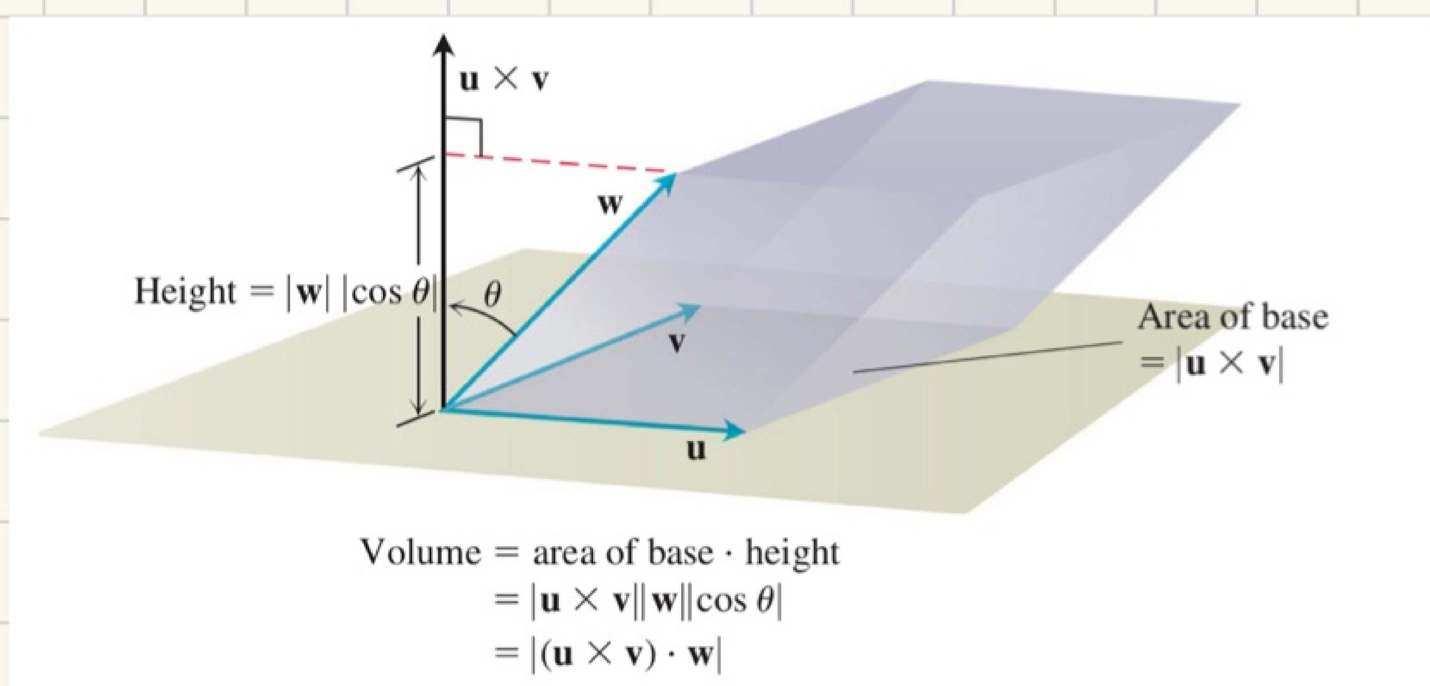


FIGURE 12.34 The number $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$ is the volume of a parallelepiped.

Def. The $(\vec{u}\times\vec{v})\cdot\vec{w}$ is called the triple scalar product of $\vec{u}$, $\vec{v}$ and $\vec{w}$ (in that order).

And we have $(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

12.5 Lines and Planes in Space

- ### Lines and Line Segments in Space

Def. Vector Equation for a Line:

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v}$ is

$\vec{r}(t) = \vec{r_0} + t\vec{v}, \quad -\infty < t < \infty$.

where r is the position vector of a point $P(x, y, z)$ on L and $\vec{r_0}$ is the position vector of $P_0(x_0, y_0, z_0)$.

Def. Parametric Equation for a Line:

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is

$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty$.

- ### The Distance from a Point to a Line in Space

Def. Distance from a Point S to a Line Through P Parallel to $\vec{v}$:

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}.$$

- ### An equation for a Plane in Space

The plane through $P_0(x_0, y_0, z_0)$ normal to $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ has

Vector equation: $\vec{n} \cdot \vec{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$.

Component equation simplified: $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

- ### The Distance from a Point to a Plane

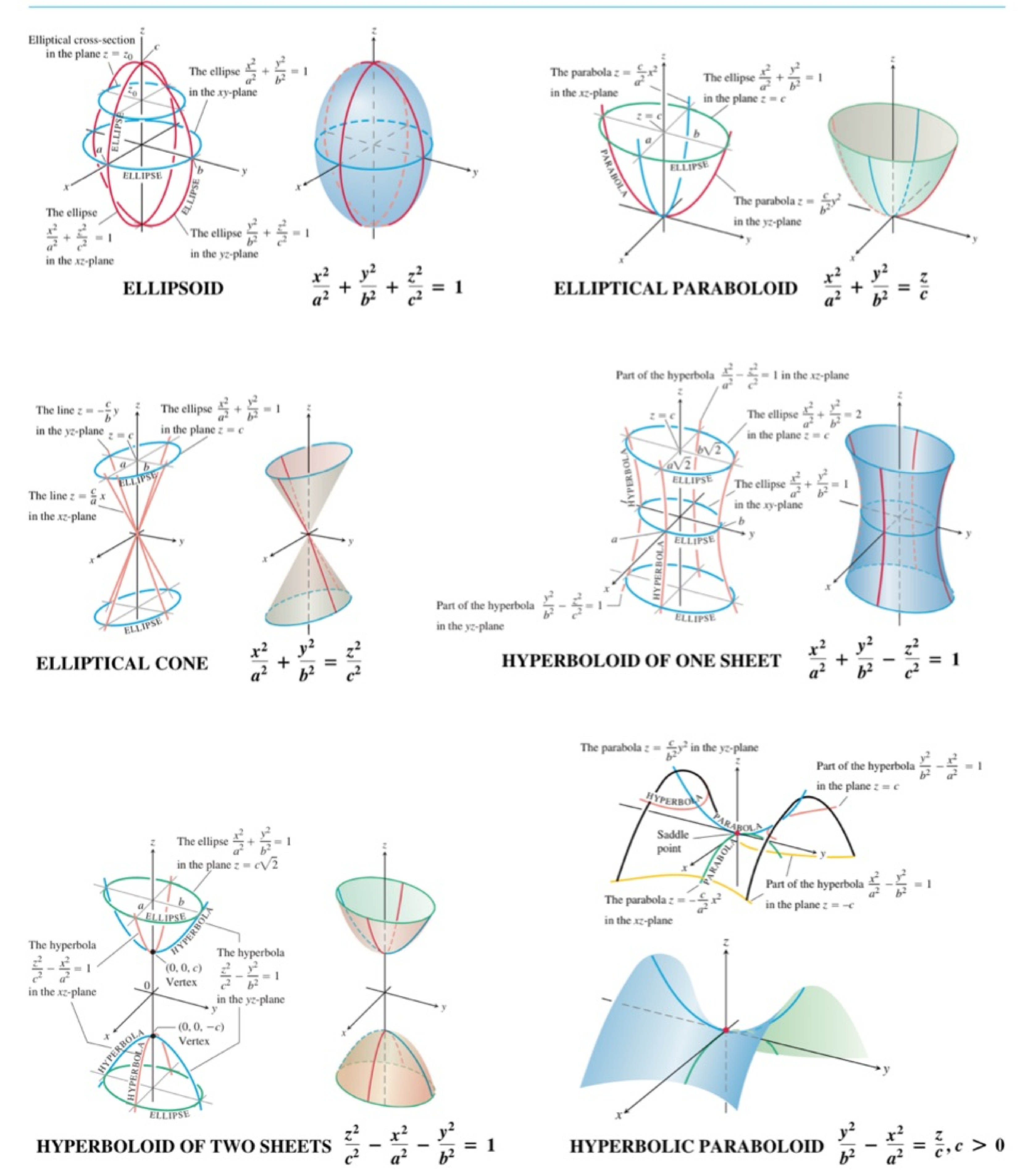
Def. If P is a point with normal $\vec{n}$, then the distance from any point S to the plane is

$$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|.$$

12.6 Cylinders and Quadratic Surfaces

As shown in the right image →

TABLE 12.1 Graphs of Quadric Surfaces



IV. Vector-Valued Functions and Motion in Space

13.1 Curves in Space and Their Tangents

· Curves in Space

Def. When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t), \quad y = g(t), \quad z = h(t), \qquad t \in I.$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the curve in space.

It can also be represented in vector form:

$$\triangle \; \vec{r}(t) = \vec{OP} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}.$$

We say f, g, h are the component function.

· Limit and Continuity

Def. Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a vector function with domain D, and L a vector. We say that $\vec{r}$ has limit L as t approaches t_0 and write

$$\triangle \; \lim_{t \to t_0} \vec{r}(t) = L.$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ with that for all $t \in D$,

$$|\vec{r}(t) - L| < \epsilon \text{ whenever } 0 < |t - t_0| < \delta.$$

Def. A vector function $r(t)$ is continuous at a point $t = t_0$ in its domain if $\lim_{t \to t_0} r(t) = r(t_0)$. The function is continuous if it is continuous over its interval domain.

· Derivatives and Motion

Def. The vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ has a derivative (is differentiable) at t if f, g, h have derivatives at t. The derivative is the vector function

$$\triangle \; \vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{df}{dt}\vec{i} + \frac{dg}{dt}\vec{j} + \frac{dh}{dt}\vec{k}.$$

The curved traced by $\vec{r}$ is smooth if $\frac{d\vec{r}}{dt}$ is continuous and never $\vec{0}$.

Def. If $\vec{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

is the particle's velocity vector, tangent to the curve. At any time t, the direction of $\vec{v}$ is the direction of motion, the magnitude of $\vec{v}$ is the particle's speed, and the derivative $\vec{a} = \frac{d\vec{v}}{dt}$, when it exists, is the particle's acceleration vector.

· Vector Functions of Constant Length

Def. If r is a differentiable vector function of t of constant length, then

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = 0.$$

13.2 Integrals of Vector Functions ; Projectile Motion

· Integrals of Vector Functions

Def. The indefinite integral of $\vec{r}$ with respect to t is the set of all antiderivatives of $\vec{r}$, denoted by $\int \vec{r}(t)dt$. If $\vec{R}$ is any antiderivative of $\vec{r}$, then

$$\int \vec{r}(t)dt = \vec{R}(t) + C .$$

Def. If the components of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ are integrable over $[a,b]$, then so is $\vec{r}$, and the definite integral of $\vec{r}$ from a to b is

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt\right)\vec{i} + \left(\int_c^b g(t)dt\right)\vec{j} + \left(\int_a^b h(t)dt\right)\vec{k}$$

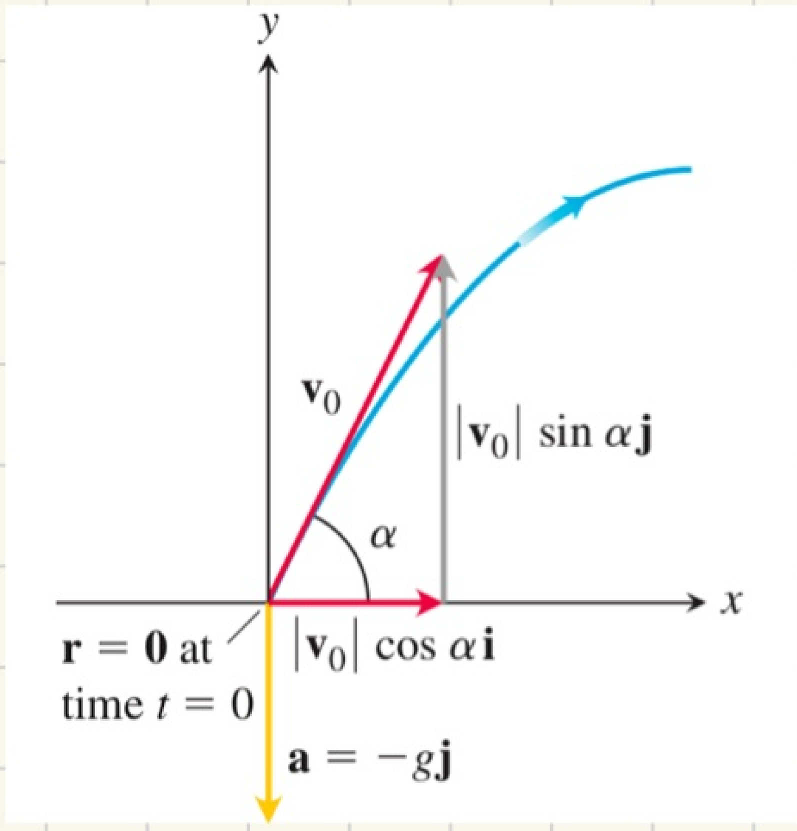
· Ideal Projectile Motion Equations

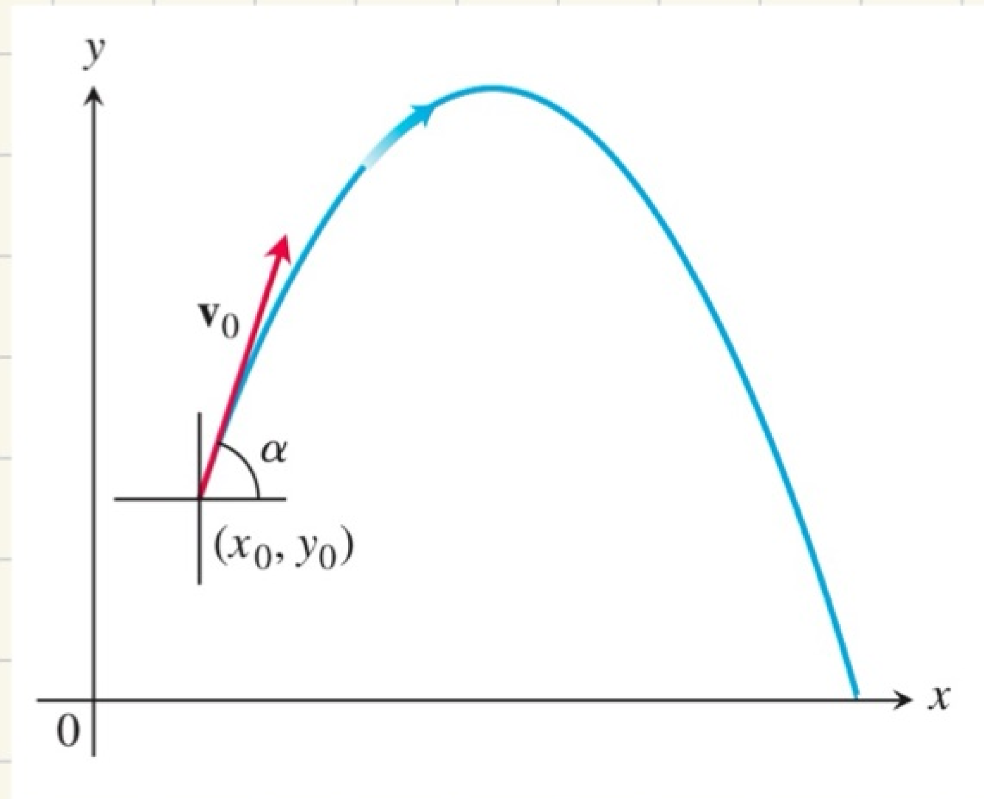
$$\vec{r} = (v_0 \cos\alpha)\vec{i} + \left((v_0 \sin\alpha)t - \tfrac{1}{2}gt^2\right)\vec{j}$$

$$y_{max} = \frac{(v_0 \sin\alpha)^2}{2g}$$

$$t = \frac{2v_0 \sin\alpha}{g}$$

$$R = \frac{v_0^2}{g}\sin 2\alpha$$





13.3 Arc Length in Space

· Arc Length Along a Space Curve

Def. The length of a smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \le t \le b$, that is traced exactly once as t increases from $t=a$ to $t=b$, is

$$\triangle\ L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |\vec{v}|\, dt$$

Arc Length Parameter with Base Point $P(t_0) = (x(t_0), y(t_0), z(t_0))$.

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau$$ where $\vec{v} = \frac{d\vec{r}}{dt}$ τ : "tau".

13.4 Curvature and Normal Vectors of a Curve

· Curvature of a Plane Curve

Def. If $\vec{T}$ is the unit vector of a smooth curve, the curvature function of the curve is

$$\circ\ \kappa = \left|\frac{d\vec{T}}{ds}\right|$$ κ : "kappa"

Formula for Calculating Curvature

If $\vec{r}(t)$ is a smooth curve, then the curvature is the scalar function

$$☆\ \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| .$$

where $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ is the unit tangent vector.

- The Principal Unit Normal Vector

Def. At a point where $\kappa \neq 0$, the principle unit normal vector for a smooth curve in the plane is

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}.$$

Formula for Calculating N.

If $\vec{r}(t)$ is a smooth curve, then the principle unit normal is

$$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

- Circle of Curvature for Plane Curves

The circle of curvature or osculating circle at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P.
3. has center that lies toward the concave or inner side of the curve

The radius of curvature of the curve at P is the radius of the circle of curvature, which is

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}.$$

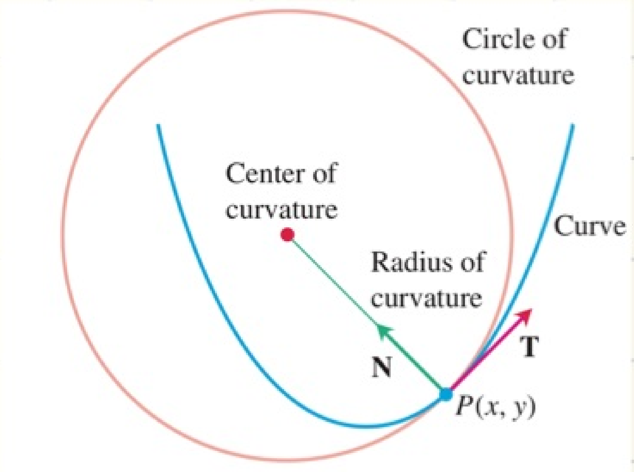


FIGURE 13.20 The center of the osculating circle at $P(x, y)$ lies toward the inner side of the curve.

- Curvature and Normal Vectors for Space Curves

If a smooth curve in space is specified by the position vector $\vec{r}(t)$ as a function of some parameter t. and if s is the arc length parameter of the curve, then the unit tangent vector $\vec{T}$ is $\frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|}$. The curvature in space is then defined to be

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

just as for plane curves. The vector $\frac{d\vec{T}}{ds}$ is orthogonal to $\vec{T}$, and we defined the principle unit normal to be

$$\vec{N} = \frac{1}{\kappa}\cdot\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}.$$

V. Partial Derivatives

14.1 Functions of Several Varibles

- Real-valued Function

Def. Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \cdots, x_n)$. A real-valued function f on D is a rule that assigns a unique (single) real number
$w = f(x_1, x_2, \cdots, x_n)$.
to each element in D. The set D is the function's domain. The set of w-values taken on by f is the function's range. The symbol w is the dependent variable of f, and f is said to be a function of the n independent variables x_1 to x_n. We also call the x_j's the function's input variables and call w the function's output variable.

- Functions of Two Variables

Def. A point (x_0, y_0) in a region (set) R in the xy-plane is an interior point of R if it is the center of a disk of positive radius that lies entirely in R. A point (x_0, y_0) is a boundary point of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lies in R.
The interior points of a region, as a set, make up the interior of the region. The region's boundary points make up its boundary. A region is open if it consists entirely of interior points. A region is closed if it contains all its boundary points.

Def. A region in the plane is bounded if it lies inside a disk of finite radius. A region is unbounded if it is not bounded.

Def. The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y) = c$ is called a level curve of f. The set of all points $(x, y, f(x,y))$ in space, for (x,y) in the domain of f, is called the graph of f. The graph of f is also called the surface $z = f(x,y)$.

P.S. Three variables are similar.

14.2 Limits and Continuity in Higher Dimensions

- Limits for Function of Two Variables

Def. We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0, y_0), write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x,y) in the domain of f,

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

Theorem: Properties of Limits of Functions of Two Variables
Same as one varible! No need to record.

- Continuity

Def. A function $f(x,y)$ is continuous at the point (x_0,y_0) if
1. f is defined at (x_0,y_0)
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists and $= f(x_0,y_0)$.

A function is continuous if it is continuous at every point of its domain

Two Path Test for Nonexistence of a Limit
If a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0,y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist. △

Continuity of Composites
If f is continuous at (x_0,y_0) and g is a single-variable function continuous at $f(x_0,y_0)$, then the composite function $h=g\circ f$ defined by $h(x,y)=g(f(x,y))$ is continuous at (x_0,y_0).

14.3 Partial Derivatives

- Partial Derivatives of a Function of Two Variables

Def. The partial derivative of $f(x,y)$ with respect to x at the point (x_0,y_0) is

△ $$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

provided the limit exists. Also noted as $f_x(x_0,y_0)$.
And respect to y is similar

- Second-Order Partial Derivatives

Def. When we differentiate a function $f(x,y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right).$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

- The Mixed Derivative Theorem

Def. If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a,b) and are all continuous at (a,b), then

○ $f_{xy}(a,b) = f_{yx}(a,b)$.

- Differentiability

Def. A function $z=f(x,y)$ is differentiable at (x_0,y_0) if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y.$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f differentiable if it is differentiable at every point in its domain, and say that its graph is a smooth surface.

Theorem: The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x,y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0,y_0). Then

$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$.

in the value of f that results from moving from (x_0,y_0) to another point $(x_0+\Delta x, y_0+\Delta y)$ in R satisfies an equation of the form

$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$.

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

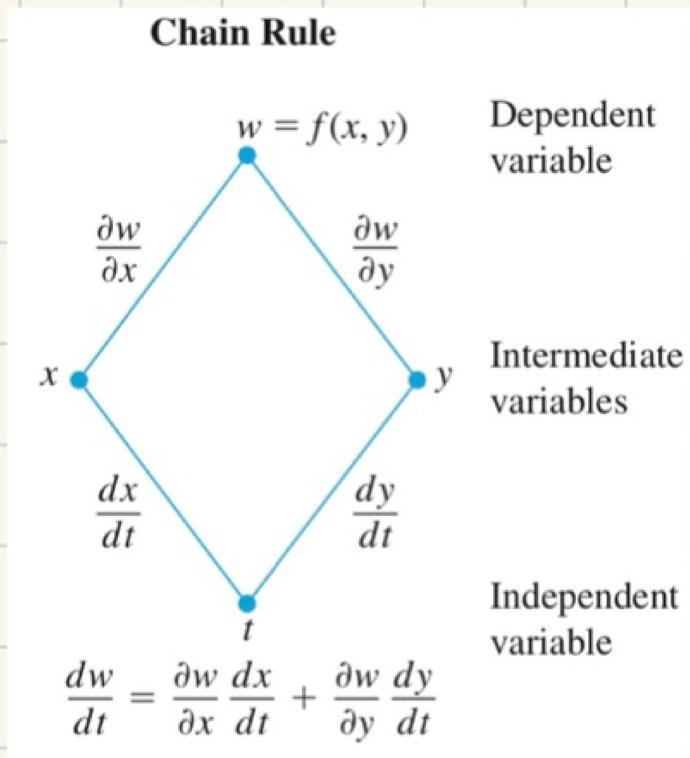
14.4 The Chain Rule

· Functions of Two Variables and Three Variables

Theorem: Chain Rule For Functions of One Independent Variable and Two Intermediate Variables

If $w=f(x,y)$ is differentiable and if $x=x(t)$, $y=y(t)$ are differentiable functions of t, then the composite $w=f(x(t), y(t))$ is a differential function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$



If $w=f(x,y,z)$ and is differentiable, then

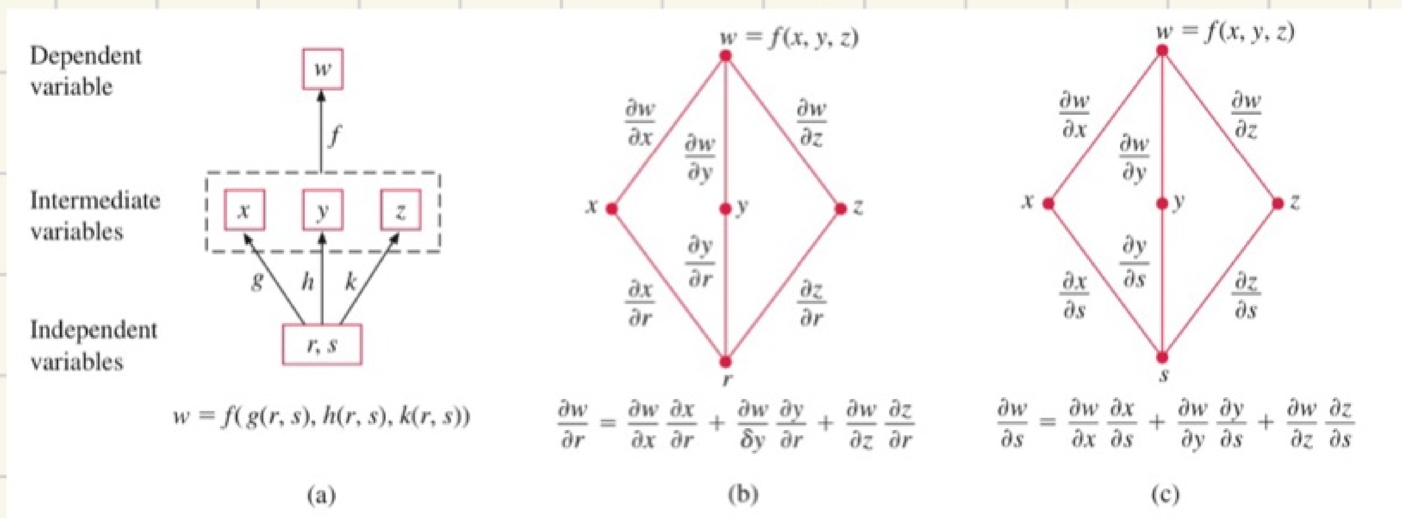
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}.$$

· Functions Defined on Surfaces

Theorem: Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that $w=f(x,y,z)$, $x=g(r,s)$, $y=h(r,s)$, and $z=k(r,s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by

$$\begin{cases} \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}. \\ \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s} \end{cases}$$



P.S. Two Intermediate Variables is similar.

· Implicit Differential Revisited

Theorem: A Formula for Implicit Differentiation

Suppose that $F(x,y)$ is differentiable and that the equation $F(x,y)=0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

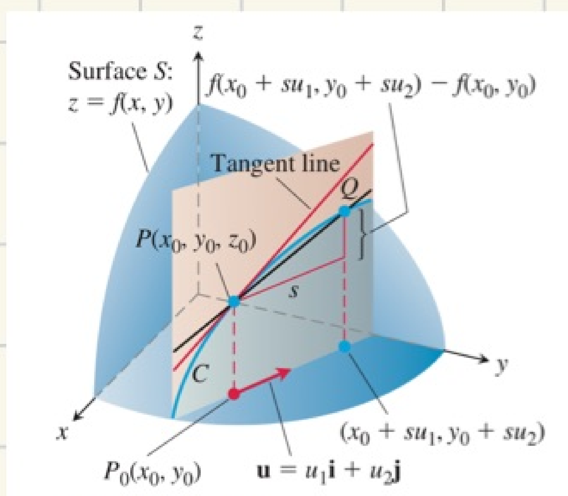


FIGURE 14.28 The slope of the trace curve C at P_0 is $\lim_{Q \to P}$ slope (PQ); this is the directional derivative $\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (D_{\mathbf{u}}f)_{P_0}$.

14.5 Directional Derivatives and Gradient Vectors

· Directional Derivatives in the Plane

Def. The derivative of f at $P_0(x_0,y_0)$ in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

provided the limit exits. Also denoted by $(D_{\vec{u}}f)_{P_0}$

Calculation and Gradients

Def. The gradient vector (gradient) of $f(x,y)$ at a point $P_0(x_0,y_0)$ is the vector

△ $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$. ∇f: "grad f". ∇: "del".

obtained by evaluating the partial derivatives of f at P_0.

Theorem: The directional derivative is a dot product.

If $f(x,y)$ is differentiable in an open region containing $P_0(x_0,y_0)$, then

△ $\left(\frac{df}{ds}\right)_{\vec{u},P_0} = (\nabla f)_{P_0}\cdot\vec{u}$. where $\vec{u}$ is a directional unit vector (unit tangent vector)

In brief, $D_{\vec{u}}f = \nabla f\cdot\vec{u}$

Properties of the Directional Derivative

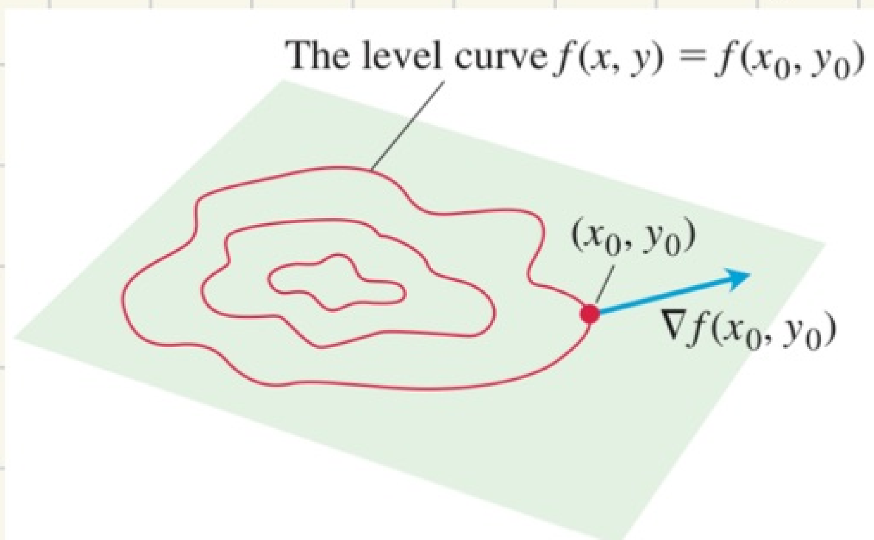
1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\vec{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
   $D_{\vec{u}}f = |\nabla f|\cos(0) = |\nabla f|$.

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\vec{u}}f = |\nabla f|\cos(\pi) = -|\nabla f|$.

3. Any direction $\vec{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\frac{\pi}{2}$ and
   $D_{\vec{u}}f = |\nabla f|\cos(\frac{\pi}{2}) = |\nabla f|\cdot 0 = 0$.

Gradients and Tangents to Level Curves

At every point (x_0,y_0) in the domain of a differentiable function $f(x,y)$, the gradient of f is normal to the level curve through (x_0,y_0).



Tangent Line to a Level Curve:

$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$.

Algebra Rules for Gradients

1. $\nabla(f\pm g) = \nabla f \pm \nabla g$. Sum & Difference Rule

2. $\nabla(kf) = k\nabla f$. Constant Multiple Rule

3. $\nabla(fg) = f\nabla g + g\nabla f$. Product Rule

4. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ Quotient Rule

Very similar to derivatives!

The Chain Rule for Paths

△ $\frac{d}{dt}f(\vec{r}(t)) = \nabla f(\vec{r}(t))\cdot\vec{r}'(t)$.

14.6 Tangent Planes and Differentials

· Tangent Plane and Normal Lines (Theorem in 12.5)

Def. The tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$

$$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0.$$

The normal line of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t.$$

The plane tangent to the surface $z = f(x,y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

· Estimating Change in a Specific Direction

Def. To estimate the change in the value of a differentiable function f when we move to a small distance ds from a point P_0 in a particular direction $\vec{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \vec{u})}_{\text{Directional derivative}} \cdot \underbrace{ds}_{\text{Distance increment}}$$

· Linearization of Function of Two Variables

Def. The linearization of a function $f(x,y)$ at a point (x_0, y_0) where f is differentiable is

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

The approximation

$$f(x,y) \approx L(x,y)$$

is the standard linear approximation of f at (x_0, y_0).

Error:

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2.$$

where M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$ and $|f_{xy}|$ on R.

· Differentials

Def. If we move from (x_0, y_0) to a point (x_0+dx, y_0+dy) nearby, the resulting change

$$df = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy.$$

in the linearization of f is called the total differential of f.

P.S. The linearization and differential of functions of more than two variables is similar!

14.7 Extreme Values and Saddle Points

· Derivative Tests for Local Extreme Values

Def. Let $f(x,y)$ be defined on a region R containing the point (a,b). Then:
$f(a,b)$ is a local maximum (minimum) value of f if $f(a,b) \geq (\leq) f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b)

Theorem: First Derivative Test for Local Extreme Values
If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain and if the first partial derivatives exists here, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

Def. An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a critical point of f.

Def. A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b) if in every disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$. The corresponding point $(a,b,f(a,b))$ on the surface $z=f(x,y)$ is called a saddle point of the surface.

Theorem: Second Derivative Test for Local Extreme Values
Suppose that $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b)=f_y(a,b)=0$. Then

i) f has a local maximum at (a,b) if $f_{xx}<0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ at (a,b).
ii) f has a local minimum at (a,b) if $f_{xx}>0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ at (a,b).
iii) f has a saddle point at (a,b) if $f_{xx}f_{yy}-f_{xy}^2<0$ at (a,b).
iv) The test is inconclusive at (a,b) if $f_{xx}f_{yy}-f_{xy}^2=0$ at (a,b). In this case, we must find some other way to determine the behavior of f at (a,b).

The expression $f_{xx}f_{yy}-f_{xy}^2$ is called the discriminant of Hessian of f.

$$f_{xx}f_{yy}-f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

※ Summary of Max-Min Test

The extreme values of $f(x,y)$ can occur only at
i) boundary points of the domain of f.
ii) critical points (interior points where $f_x=f_y=0$ or DNE).

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a,b) and $f_x(a,b)=f_y(a,b)=0$, the nature of $f(a,b)$ can be tested with the Second Derivative Test:
i) $f_{xx}<0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ at (a,b) => loc. max.
ii) $f_{xx}>0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ at (a,b) => loc. min.
iii) $f_{xx}f_{yy}-f_{xy}^2<0$ at (a,b) => Saddle point
iv) $f_{xx}f_{yy}-f_{xy}^2=0$ at (a,b) => test is inconclusive

14.8 Lagrange Multipliers

- The Method of Lagrange Multipliers

Theorem: The Orthogonal Gradient Theorem

Suppose that $f(x,y,z)$ is differentiable in a region whose interior contains a smooth curve

$C: \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

If P_0 is a point on C where f has a local maximum or minimum relative to its value on C, then ∇f is orthogonal to C at P_0.

Collary: At the point on a smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ where a differentiable function $f(x,y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot v' = 0$.

Def. Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq 0$ when $g(x,y,z)=0$.

To find the local maximum and minimum values of f subject to the constraint $g(x,y,z)=0$, find the values of x, y, z, and λ that simultaneously satisfy the equations

$\nabla f = \lambda \nabla g$ and $g(x,y,z) = 0$.

- Lagrange Multipliers with Two Constraints

?

14.9 Taylor's Formula for Two Variables

- Taylor's Formula for $f(x,y)$ at the Point (a,b).

Suppose $f(x,y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centered at a point (a,b). Then, throughout R,

$$f(a+h,b+k) = f(a,b) + (hf_x + kf_y)\big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\big|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)} + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(a+ch,\, b+ck)}.$$

At the origin is $(a,b) = (0,0)$.

V. Multiple Integrals

15.1 Double and Iterated Integrals over Rectangles

· Double Integrals

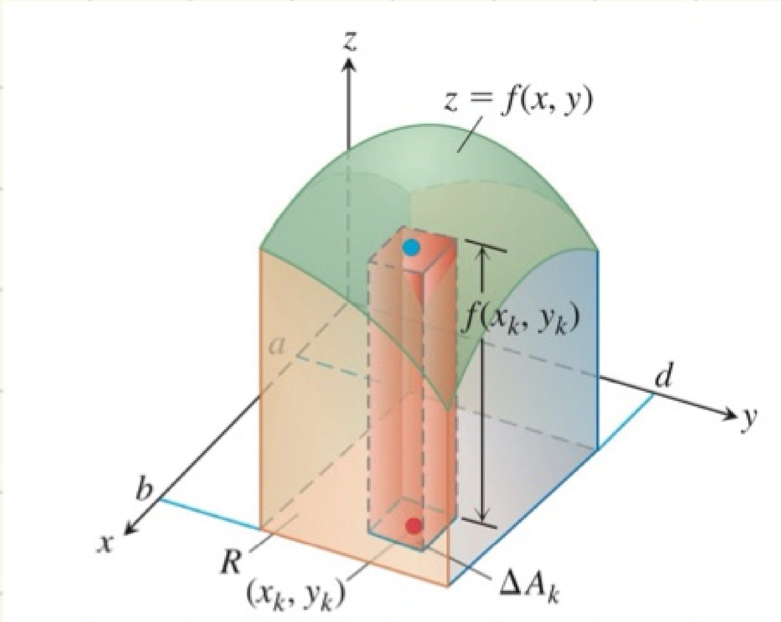
Def. Suppose a function $f(x,y)$ defined on R: $a\le x\le b$, $c\le y\le d$, then

$$\triangle \iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k .$$

where $\Delta A = \Delta x \cdot \Delta y$.

As volumes:

$$\text{Volume} = \iint_R f(x,y)\,dA .$$



· Fubini's Theorem

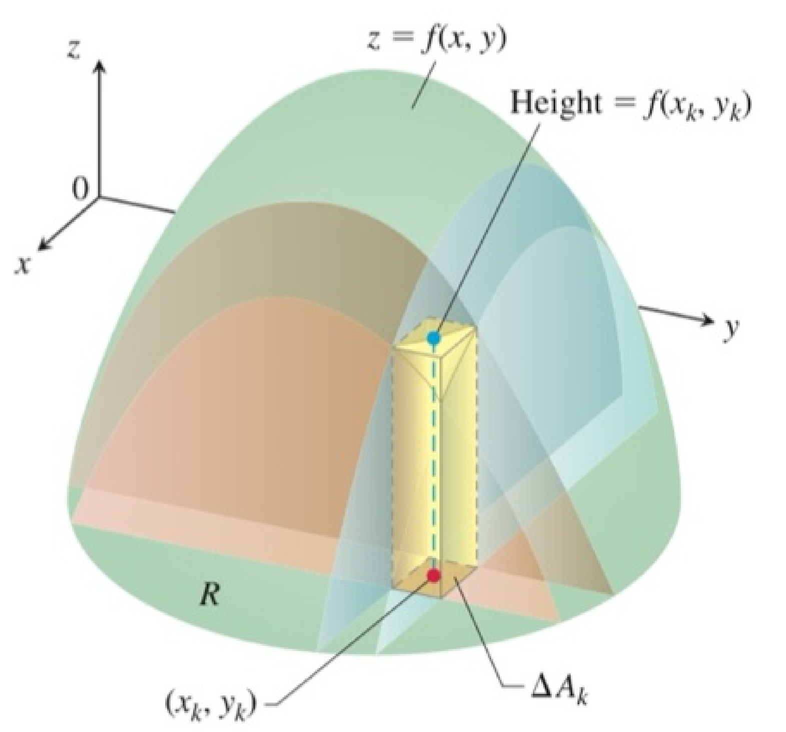
Theorem: Fubini's Theorem (First Form)

If $f(x,y)$ is continuous throughout the rectangular region R: $a\le x\le b$, $c\le y\le d$, then

$$\triangle \iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx .$$

15.2 Double Integrals over General Regions

· Volumes of General Solid Regions



Volume $= \lim \sum f(x_k, y_k)\,\Delta A_k = \iint_R f(x,y)\,dA$

FIGURE 15.9 We define the volumes of solids with curved bases as a limit of approximating rectangular boxes.

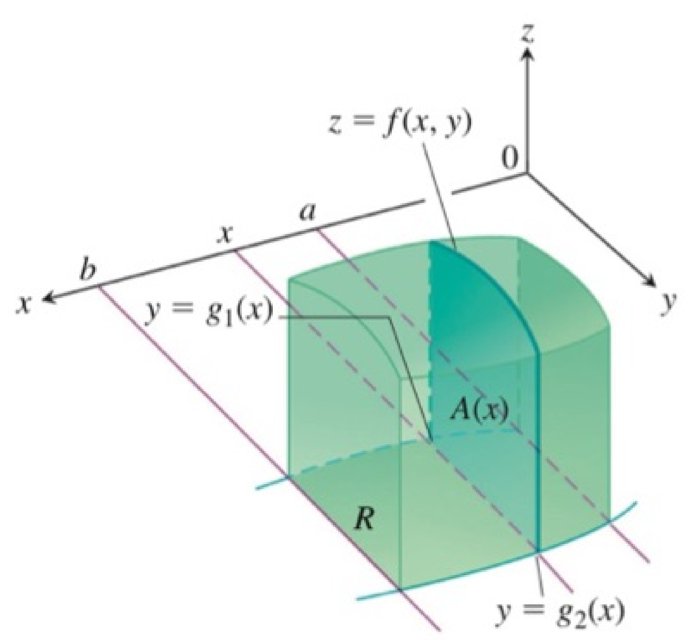


FIGURE 15.10 The area of the vertical slice shown here is $A(x)$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$:

$$\int_a^b A(x)\,dx = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

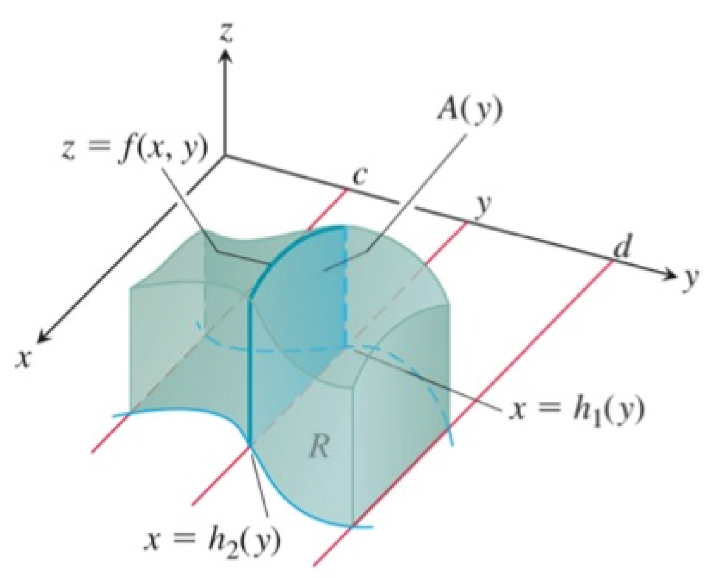


FIGURE 15.11 The volume of the solid shown here is

$$\int_c^d A(y)\,dy = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

For a given solid, Theorem 2 says we can calculate the volume as in Figure 15.10, or in the way shown here. Both calculations have the same result.

· Fubini's Theorem (Stronger Form)

Theorem:

Let $f(x,y)$ be continuous on a region R.

1. If R is defined by $a\le x\le b$, $g_1(x)\le y\le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then
$$\iint_R f(x,y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx .$$

2. If R is defined by $c\le y\le d$, $h_1(y)\le x\le h_2(y)$, with h_1 and h_2 continuous on $[c,d]$, then
$$\iint_R f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy .$$

- Properties

If $f(x,y)$ and $g(x,y)$ are continuous on the bounded region R, then the following properties hold:

1. $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$ $(c\in R)$ Constant Multiple

2. $\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$. Sum and Difference

3. (a) $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R

Domination

(b) $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R

4. $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$ Additivity

if R is the union of two nonoverlapping regions R_1 and R_2.

15.3 Area by Double Integration

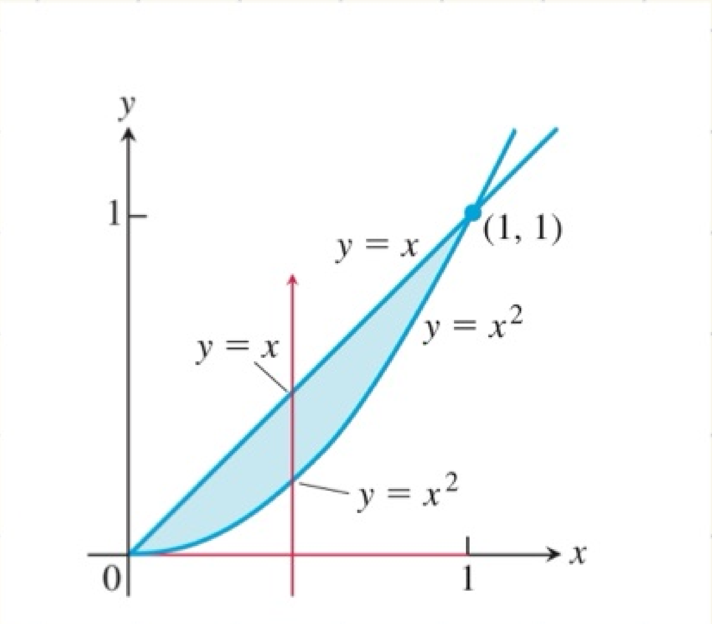
- Areas of Bounded Regions

Def. The area of a closed, bounded plane region R is

$$A = \iint_R dA.$$

e.g. Find the area of region R bounded by $y=x$ and $y=x^2$ in the first quadrant.

$$\begin{aligned} A &= \iint_R dA \\ &= \int_0^1\int_{x^2}^{x} dy\,dx \\ &= \int_0^1 [y]_{x^2}^{x}\,dx \\ &= \int_0^1 (x-x^2)\,dx = \frac{1}{6}. \end{aligned}$$



- Average Value

Def. The average value of f over R is

$$\frac{1}{\text{area of } R}\iint_R f\,dA.$$

15.4 Double Integrals in Polar Form

- Area in Polar Coordinates

$$A = \iint_R r\,dr\,d\theta.$$

- Changinging Cartesian into Polar

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

15.5 Triple Integrals in Rectangular Coordinates

· **Triple Integrals**

Def. Suppose a function $F(x,y,z)$ on D, then:

△ $$\iiint_D F(x,y,z)\,dV = \iiint_D F(x,y,z)\,dx\,dy\,dz = \lim_{n\to\infty}\sum_{k=1}^{n} F(\Delta x_k, \Delta y_k, \Delta z_k)\,\Delta V_k.$$

· **Volume of a Region in Space**

Def. The volumn of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

· **Average Value of a Function in Space**

Def. The average value of F over D is

$$\frac{1}{\text{volume of } D}\iiint_D F\,dV.$$

P.S. Triple integrals have the same algebraic properties as double and single integrals.

15.6 Moments and Centers of Mass

· **Mass and First Moments**

Def. Suppose $\delta(x,y,z)$ is the density (mass/unit volume) of an object occupying a region D in space, we have:

△ Mass: $M = \lim_{n\to\infty}\sum_{k=1}^{n}\Delta m_k = \lim_{n\to\infty}\sum_{k=1}^{n}\delta(x_k, y_k, z_k)\,\Delta V_k = \iiint_D \delta(x,y,z)\,dV.$

First moments about the coordinate planes:

$M_{yz} = \iiint_D x\delta\,dV, \quad M_{xz} = \iiint_D y\delta\,dV, \quad M_{xy} = \iiint_D z\delta\,dV.$

Three dimensional Solid

Center of Mass:

$\bar{x} = \dfrac{M_{yz}}{M}, \quad \bar{y} = \dfrac{M_{xz}}{M}, \quad \bar{z} = \dfrac{M_{xy}}{M}$

Suppose $\delta(x,y)$ is the density at (x,y)

Mass: $M = \iint_R \delta\,dA$

Two-dimensional Plate

First moments: $M_y = \iint_R x\delta\,dA, \quad M_x = \iint_R y\delta\,dA$

Center of Mass: $\bar{x} = \dfrac{M_y}{M}, \quad \bar{y} = \dfrac{M_x}{M}.$

Well... Actually 15.6 is being skipped but IDK...

15.7 Triple Integrals in Cylindrical and Spherical Coordinates

· Integration in Cylindrical Coordinates

Def. Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which $r \geq 0$.

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate

Equations:

$$\begin{cases} x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \\ r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x} \end{cases}$$

Integral of $f(r, \theta, z)$

$$\iiint_D f\,dV = \iiint_D f\,dz\,r\,dr\,d\theta$$

$$\iiint_D f(r,\theta,z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

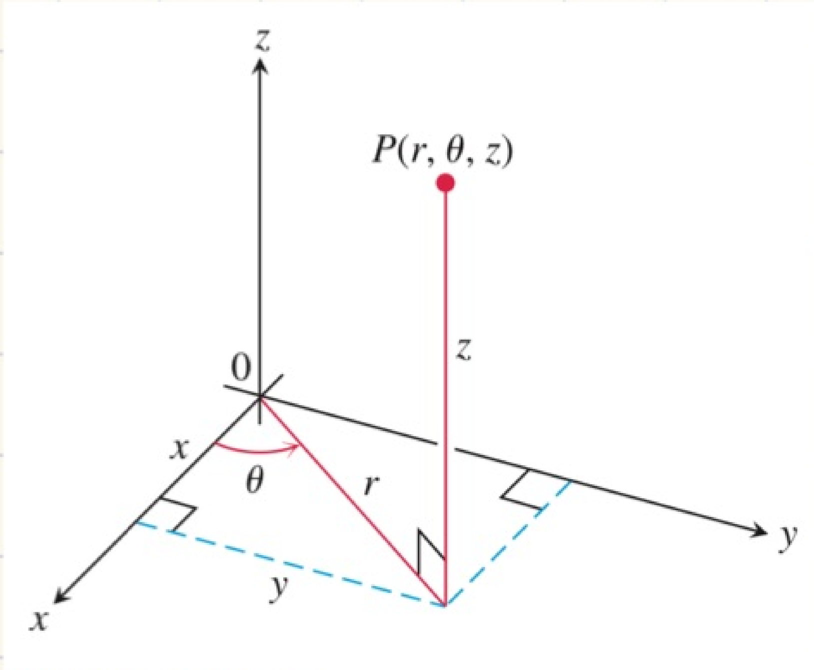
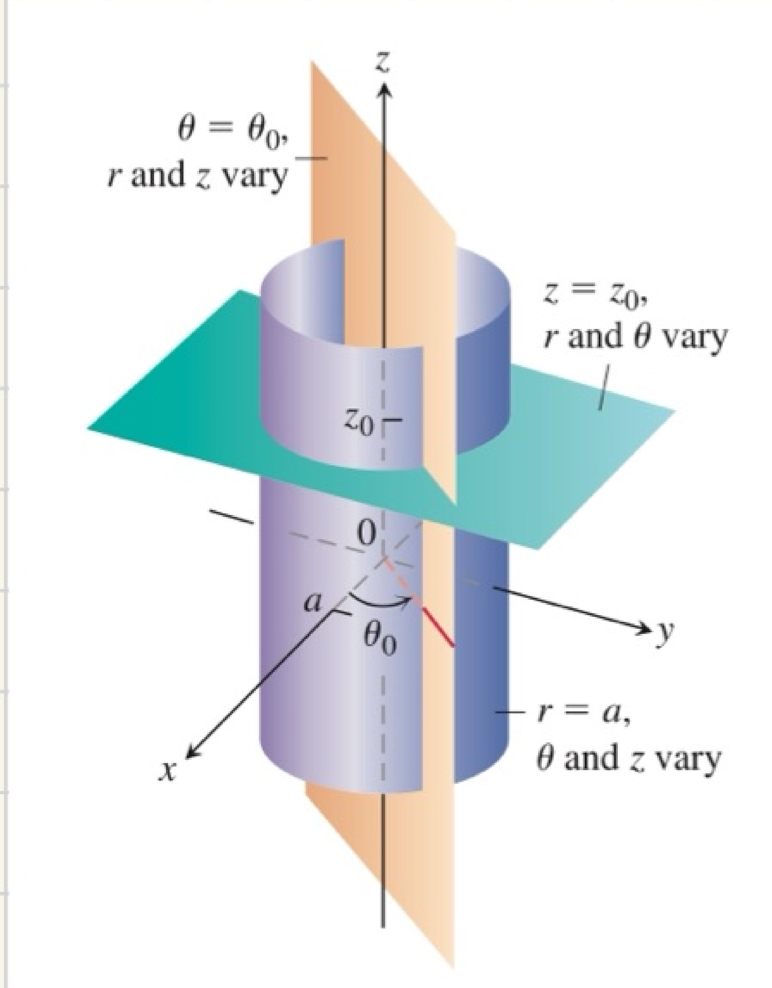


FIGURE 15.43 The cylindrical coordinates of a point in space are r, θ, and z.



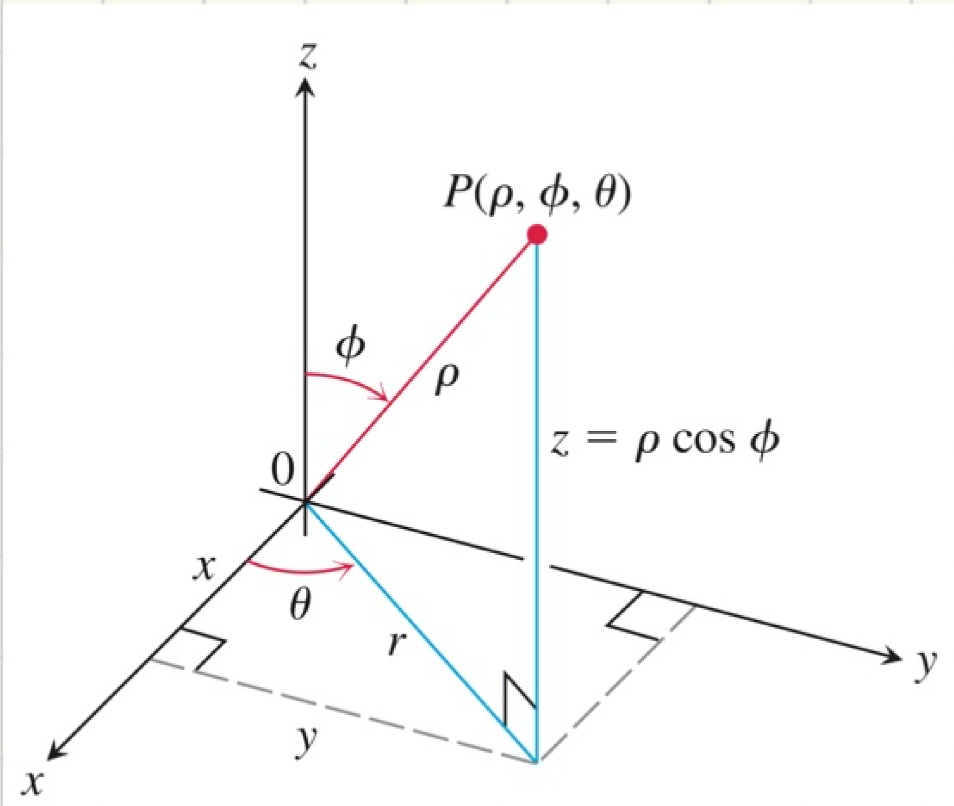
· Spherical Coordinates and Integration

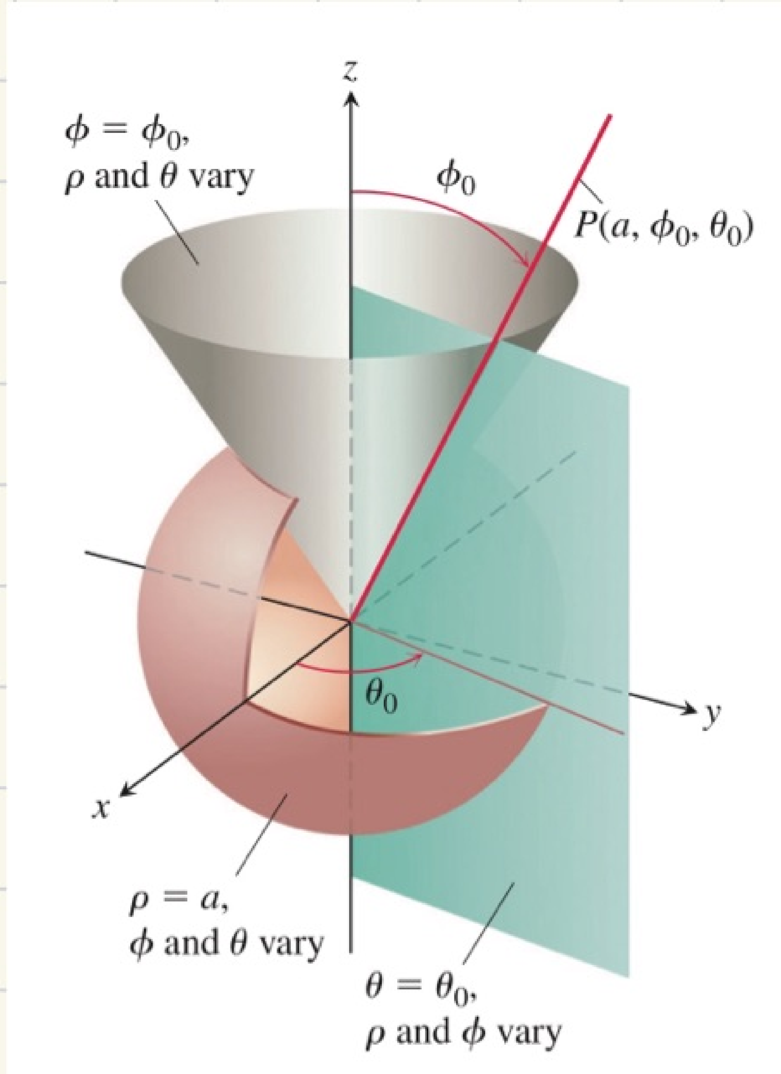
Def. Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin $(\rho \geq 0)$
2. ϕ is the angle $\vec{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$
3. θ is the angle from cylindrical coordinates.

Equations:

$$\begin{cases} r = \rho\sin\phi, \quad x = r\cos\theta = \rho\sin\phi\cos\theta \\ z = \rho\cos\phi, \quad y = r\sin\theta = \rho\sin\phi\sin\theta \\ \rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2} \end{cases}$$





Integral of $f(\rho, \phi, \theta)$:

$$\iiint_D f(\rho,\phi,\theta)\,dV = \iiint_D f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\iiint_D f(\rho,\phi,\theta)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{\phi=\phi_{min}}^{\phi=\phi_{max}}\int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

15.8 Substitutions in Multiple Integrals

Subtitutions in Double Integrals

Def. The Jacobian determinant or Jacobian of the coordinate transformation $x=g(u,v)$, $y=h(u,v)$ is

$$J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x \partial y}{\partial u \partial v} - \frac{\partial y \partial x}{\partial u \partial v} = \frac{\partial (x,y)}{\partial (u,v)}$$

Theorem : Subtitution for Double Integrals

Suppose that $f(x,y)$ is continuous over the region R. Let G be preimage of R under the transformation $x=g(u,v)$, $y=h(u,v)$, assumed to be one-to-one on the interior of G.
If the functions g and h have continuous first partial derivatives with the interior of G, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

Subtitutions in Triple Integrals

Def. Similar to triple integrals with equation:

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G H(u,v,w)\,|J(u,v,w)|\,du\,dv\,dw.$$

where $J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$.

VI. Integrals and Vector Fields

16.1 Line Integrals

· Line Integrals

Def. If f is defined on a curve C given parametrically by $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$, $a \le t \le b$.
Then the line integral of f over C is

$$\triangle \int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k = \int_a^b f(g(t), h(t), k(t))\,|\vec{v}(t)|\,dt.$$

provided this limit exists

· Additivity

Def. If a piecewise smooth curve C is made by joining a finite number of smooth curves $C_1, C_2, \cdots, C_n$ end to end, then:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds.$$

The value of the line integral along a path joining two points can change if you change the path between them.

· Mass and Moment Calculations

Def. Suppose a smooth curve C in space, $\delta(x,y,z)$ is the desity at (x,y,z).

Mass: $M = \int_C \delta\,ds$

First moments about the coordinate planes:

$$M_{yz} = \int_C x\,\delta\,ds, \quad M_{xz} = \int_C y\,\delta\,ds, \quad M_{xy} = \int_C z\,\delta\,ds.$$

Coordinates of the center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}.$$

Moments of inertia about axes and other lines:

$$I_x = \int_C (y^2+z^2)\,\delta\,ds, \quad I_y = \int_C (x^2+z^2)\,\delta\,ds, \quad I_z = \int_C (x^2+y^2)\,\delta\,ds.$$

$I_L = \int_C r^2\,\delta\,ds$ where $r(x,y,z)$ is the distance from the point (x,y,z) to line L.

· Line Integrals in the Plane