MAT

MAT1001 Calculus I
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How to learn Maths?
1. Definition/Motivation.

I. Limits and Continuity.

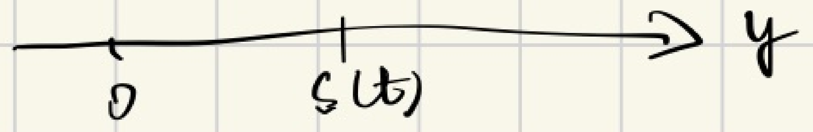
- Very rough def: We say $f(x)$ has limit L as x towards to C, if $f(x)$ towards to L, as X towards to C.
  write $f(x) \to L$, as $x \to C$.

$$\lim_{x\to c} f(x) = L$$

e.g. $\lim_{x\to 0} (2024x+328) = 328$.

$\lim_{x\to 0} \frac{2024x+328}{215+x^2} = \frac{328}{215}$.

- Motivations 1. Instantaneous speed of moving object.
  e.g. object moving on y-axis, in positive direction.



$y = s(t) = t^2$.

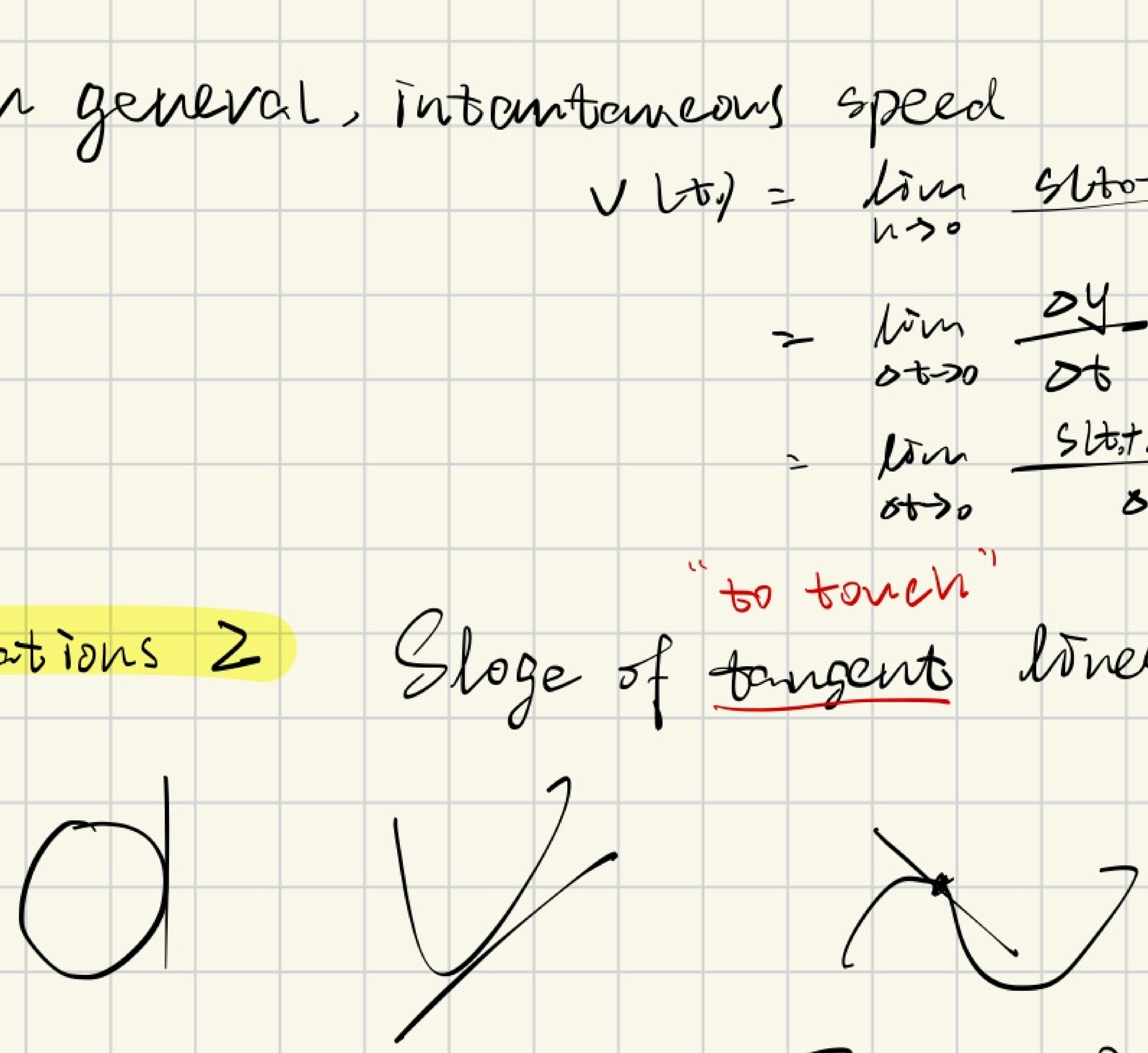
Q: Instantaneous speed $v(t)$ at $t=1$?
Idea: Consider small time interval $[1, 1+h]$ if $h>0$, $[1+h, 1]$ if $h<0$.
Average speed in time interval $[1, 1+h)$ $= \frac{\text{distance}}{\text{time duration}} = \frac{s(1+h)-s(1)}{h}$

$$\Rightarrow v(t) = \lim_{h\to 0} \frac{s(1+h)-s(1)}{h}$$
$$= \lim_{h\to 0} \frac{(1+h)^2-1^2}{h}$$
$$= \lim_{h\to 0} \frac{(1+h-1)(1+h+1)}{h}$$
$$= \lim_{h\to 0} (2+h) = 2.$$

In general, instantaneous speed

$$v(t_0) = \lim_{h\to 0} \frac{s(t_0+h)-s(t_0)}{h} \begin{matrix}\to \text{change in } y. \\ \to \text{Change in } t.\end{matrix}$$
$$= \lim_{\Delta t\to 0} \frac{\Delta y}{\Delta t}$$
$$= \lim_{\Delta t\to 0} \frac{s(t_0+\Delta t)-s(t_0)}{\Delta t} = \lim_{t\to t_0} \frac{s(t)-s(t_0)}{t-t_0}.$$

- Motivations 2 Slope of tangent lines of curves.
  "to touch"

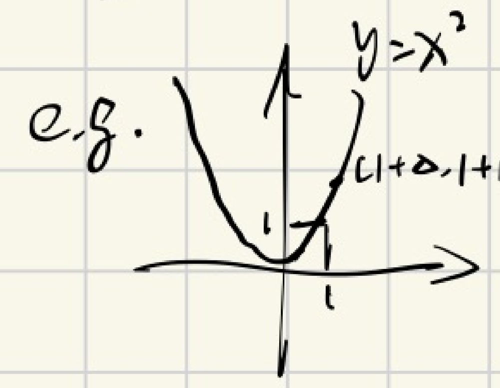


T-lines.

Q: Why care about T-lines?

A: T-line is the best line to approximate a curve.
(proof.)

1700AB Newton's method of calculating slope.

e.g. 

T-line at $(1,1)$? let m be the slope.

$$\begin{cases} 1+m\cdot o=(1+o)^2 \\ 1=1^2 \end{cases} \Rightarrow \begin{aligned} 1+m\cdot o &= o^2+2o+1 \\ m\cdot o &= o^2+2o \\ m &= o+2. \end{aligned}$$

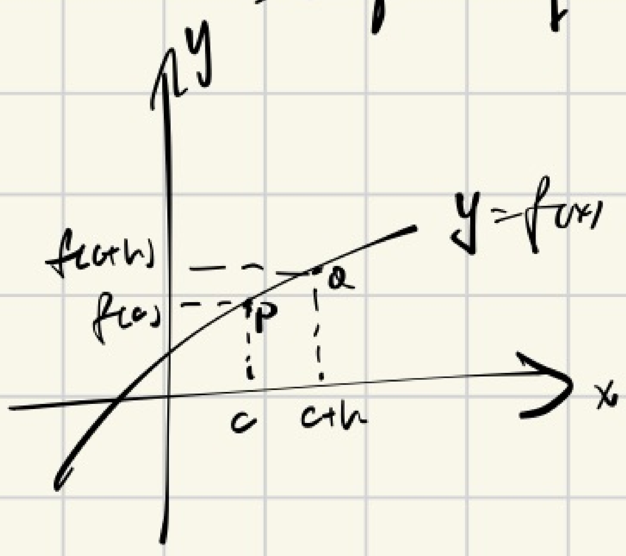
Newton let $o=0$. $\Rightarrow m=2$.

At that time Newton called "o" the "infinitisimal"

Leibnitz: T-line is the line through a pair of infinitely close points on curve

△ Rigorous def of T-line $= \lim\limits_{Q\to P}$ secant line

$\Rightarrow$ slope T-line $= \lim\limits_{Q\to P}$ slope of secant lines



$$\text{slope} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0}\frac{\Delta y}{\Delta x}$$

· Motivations 3

Let $y=f(x)$ be general function,

average rate of change of f over $[x_1, x_2] = \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

instantaneous rate of change of f at $x_1 = \lim\limits_{x_2\to x_1}\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

$$= \lim_{h\to 0}\frac{f(x_1+h)-f(x_1)}{h}$$

· More formal def of limits.

△ Let $f(x)$ be defined in an open interval containing C (a truncated neibourhood of C.), possibly not at C itself.

If $f(x)$ is arbitrarily close to the # L, as close as we wish, as long as x is close (but not equal) to C enough, that we say f approaches the limit L as $x\to C$.

Notation $\lim\limits_{x\to C} f(x)=L$. in general, has nothing to do $f(C)$!

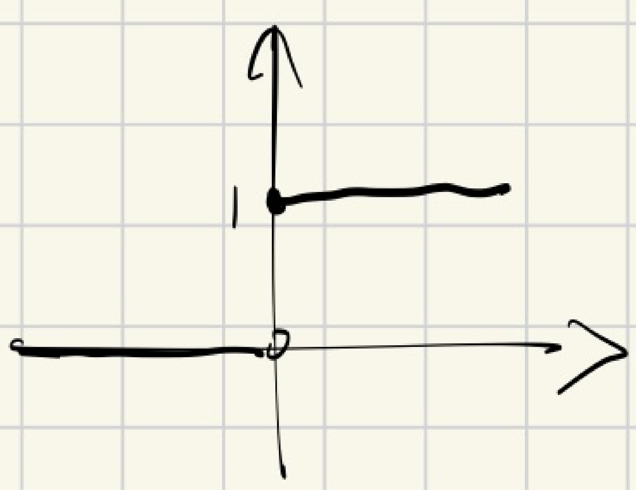
Q: Does $\lim\limits_{x\to C} f(x)$ always exist?

A: No!

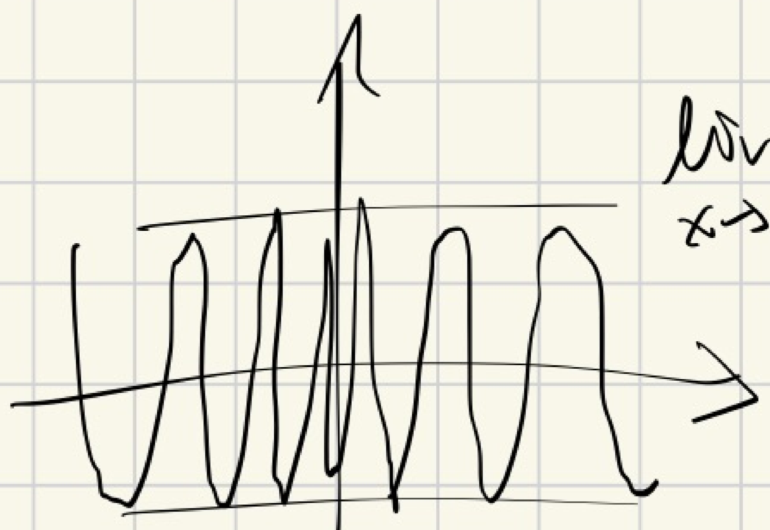
e.g. 1 $f(x)=\begin{cases} 0, & \text{if } x<0 \\ 1, & \text{if } x\geq 0 \end{cases}$

e.g. 2 $f(x)=\sin\left(\frac{1}{x}\right)$

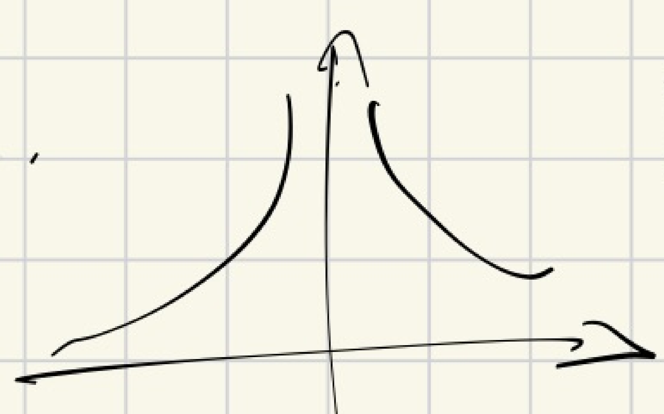
e.g. 3 $f(x)=\frac{1}{x^2}$



$\lim\limits_{x\to 0} f(x)$ DNE



$\lim\limits_{x\to 0} f(x)$ DNE.



DNE!

Limit Laws

$\lim_{x\to c} f(x)$ to $\lim_{x\to c} g(x)$ exist.

Then (i) $\lim_{x\to c}(f(x)\pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$.

(ii) k const,

$\lim_{x\to c}(kf(x)) = k\lim_{x\to c} f(x)$.

c_1, c_2 const,

$\lim_{x\to c}(c_1 \cdot f(x) + c_2 \cdot f(x)) = c_1 \cdot \lim_{x\to c} f(x) + c_2 \cdot \lim_{x\to c} f(x)$.

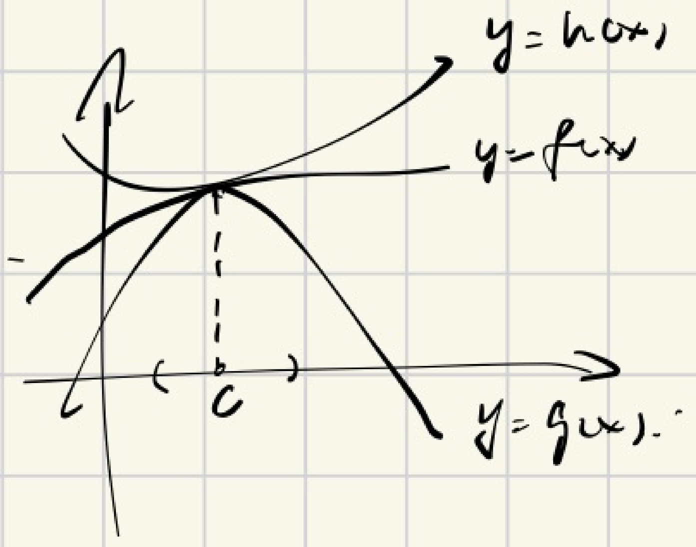
(iii) $\lim_{x\to c}(f(x)\cdot g(x)) = (\lim_{x\to c} f(x))\cdot(\lim_{x\to c} g(x))$.

(iv) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$.

(v) $\lim_{x\to c}(f(x))^n = (\lim_{x\to c} f(x))^n$

$\lim_{x\to c}(f(x))^{\frac{1}{n}} = (\lim_{x\to c} f(x))^{\frac{1}{n}}$

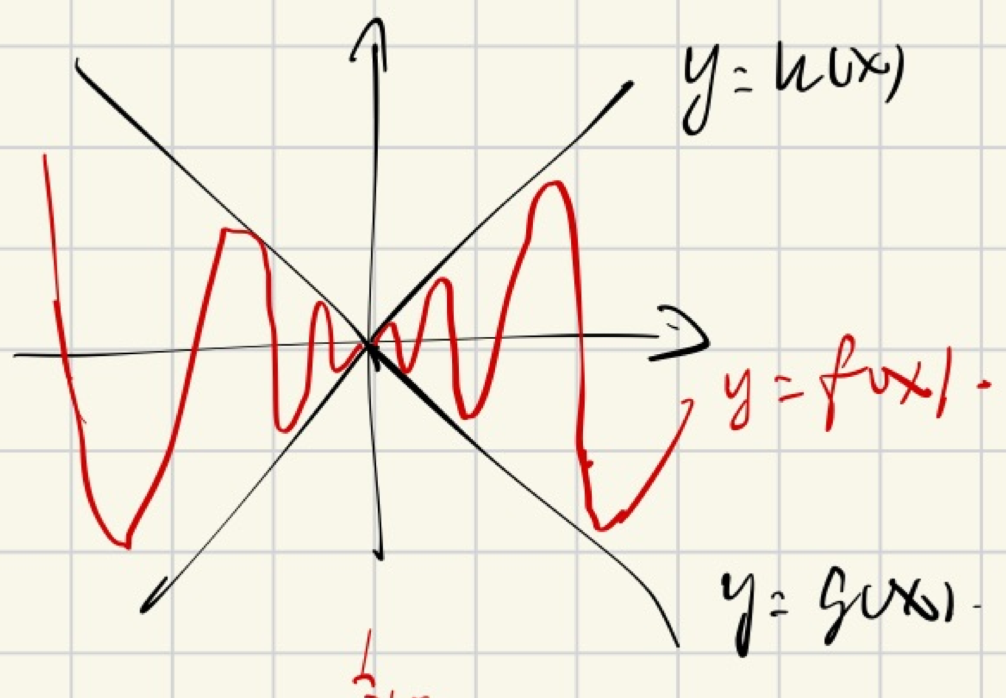
Sandwich / Squeezing / Pinching Theorem.



Suppose $g(x) \le f(x) \le h(x)$, $\forall x$ in truncated nbhd of C.
and $\lim_{x\to c} g(x)$ & $\lim_{x\to c} f(x)$ exist & $= L$.
Then, $\lim_{x\to c} f(x)$ exist & $= L$.

e.g. $f(x) = \begin{cases} x\cdot\sin(\frac{1}{x}), & x\neq 0 \\ 328, & x = 0 \end{cases}$ $\quad \lim_{x\to 0} f(x)$?

A: $\underbrace{-|x|}_{g(x)} \le -|x|\cdot|\sin\frac{1}{x}| \le \underbrace{x\sin\frac{1}{x}}_{f(x)} \le |x|\cdot|\sin\frac{1}{x}| \le \underbrace{|x|}_{h(x)}$. $(x\neq 0)$.



$\because \lim_{x\to 0}|x| = \lim_{x\to 0} = 0$

$\therefore$ According to Sandwich Theorem,

$\lim_{x\to 0} f(x) = 0$

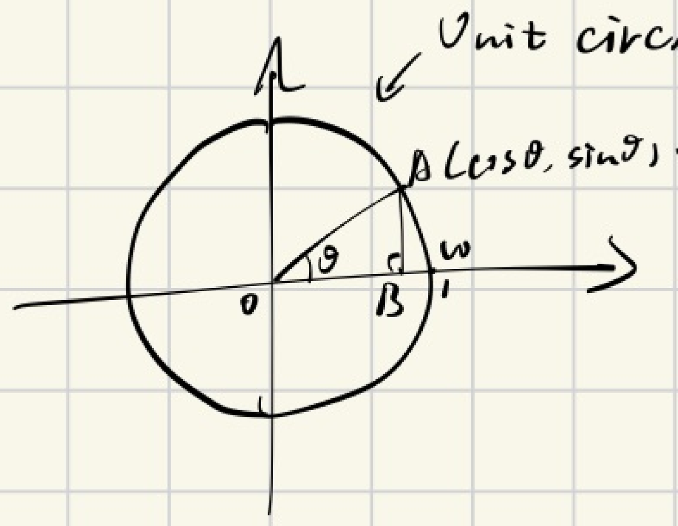
e.g. Given: $\frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos x}{x^2} < \frac{1}{2}$, $x \approx 0$. $x\neq 0$.

$\frac{1}{2}$ $\to \frac{1}{2}$ So $\frac{1}{2}$, as $x\to 0$.

then $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$

$f(x) > g(x) \Rightarrow \lim_{x\to c} f(x) \ge \lim_{x\to c} g(x)$.

e.g. $\lim\limits_{\theta\to 0}\sin\theta$? $\lim\limits_{\theta\to 0}\cos\theta$?



$AB^2 + BW^2 = AW^2 \le \widehat{AW}^2$

$(\sin\theta)^2 + (1-\cos\theta)^2 \le \theta^2$

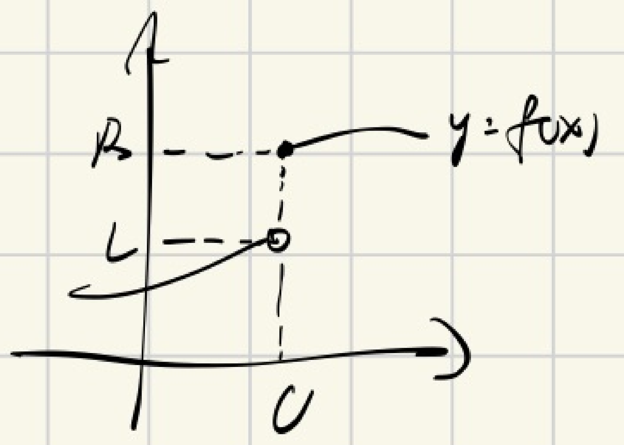
$\Rightarrow (\sin\theta)^2 \le \theta^2, \quad (1-\cos\theta)^2 \le \theta^2$

$\Rightarrow |\sin\theta| \le |\theta|, \quad |1-\cos\theta| \le |\theta|$

$-\theta \le \sin\theta \le \theta, \quad -\theta \le 1-\cos\theta \le \theta.$

$\Rightarrow \lim\limits_{\theta\to 0}\sin\theta = 0, \quad \lim\limits_{\theta\to 0}\cos\theta = 1.$

• One-sided Limits



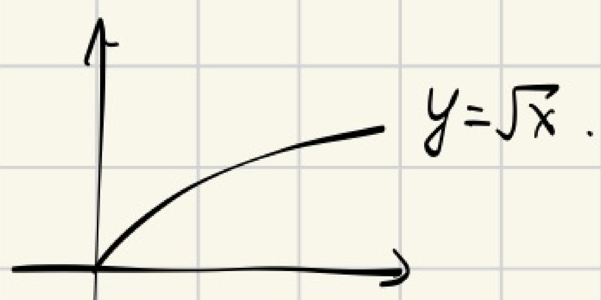
two-sided limit

$\lim\limits_{x\to c} f(x)$ DNE

Def. If $f(x) \to$ some R, as $x \to c$ in the fashion s.t. x remains $> c$, then we say $f(x)$ has right-hand limit as $x \to c$.

Write $\lim\limits_{x\to \boxed{c^+}} f(x) = R$. (left-hand limit: $\lim\limits_{x\to \boxed{c^-}} f(x) = L$)

e.g. $\lim\limits_{x\to 0^+}\sqrt{x} = 0$.



* If the function's one-side is defined while the other is not. We can have $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c^+} f(x) \,/\, \lim\limits_{x\to c^-} f(x)$.

e.g. $\lim\limits_{x\to 0}\sqrt{x} = 0$.

Q: Relationship between one-sided & two-sided limits?

A: $\lim\limits_{x\to c} f(x)$ exists if and only if $\lim\limits_{x\to c^+} f(x) = \lim\limits_{x\to c^-} f(x)$. *

From what I understand, the limit as x approaches 0 of $\sqrt{x}$ is in fact 0. I think I understand where your confusion comes from. If I understand correctly then you will have been taught that

$$\lim_{x\to 0^-}\sqrt{x} = \lim_{x\to 0^+}\sqrt{x}$$

then

$$\lim_{x\to 0}\sqrt{x}$$

will exist or in other words, both the left and right side limit *must* exist for you to find both limits. However, the subtle "thing" one must understand is that the function *has to be* defined in the negative spectrum for x, i.e. it must exist in the negative domain to begin with. The square root function isn't defined in the negative domain and therefore, by definition, you *cannot* take the limit in that domain. This doesn't mean that the limit does not exist there, it just means that you cannot take a limit there to begin with. Thus, the limit at 0 of the $f(x) = \sqrt{x}$ is in fact 0 i.e.

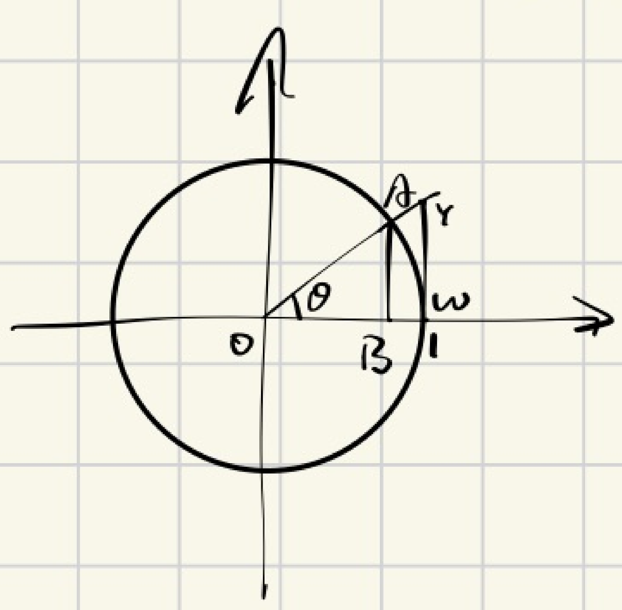
$$\lim_{x\to 0}\sqrt{x} = 0$$

• Impact Limit $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$

Idea: $\dfrac{\sin\theta}{\theta}$ is an even function

⇓

$\lim\limits_{\theta\to 0^+} = \lim\limits_{\theta\to 0^-}$ if exist.



area of $\triangle OAW = \sin\theta \cdot 1 \cdot \frac{1}{2}$

$\wedge$

area of sector $OAW = \frac{1}{2}\theta$

$\Rightarrow \frac{1}{2}\sin\theta \le \frac{1}{2}\theta \Rightarrow \dfrac{\sin\theta}{\theta} \le 1.$

area of $\triangle OWY = \tan\theta \cdot 1 \cdot \frac{1}{2}$

$\Rightarrow \frac{1}{2}\tan\theta \ge \frac{1}{2}\theta \Rightarrow \dfrac{\sin\theta}{\theta} \ge \cos\theta.$

$\Rightarrow \cos\theta \le \dfrac{\sin\theta}{\theta} \le 1.$

as $\theta \to 0^+$, $\cos\theta = 1$, $1 = 1$. Squeezing

thus $\lim\limits_{x\to 0}\dfrac{\sin\theta}{\theta} = 1.$

e.g. $\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\theta}$

$= \lim_{\theta\to 0} \left(\frac{\sin\theta}{\theta}\right) \cdot \left(\frac{1}{\cos\theta}\right)$

$= 1 \times \frac{1}{1} = 1$

e.g. $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0} \frac{1-1+2\sin^2\frac{\theta}{2}}{\theta}$

$= \lim_{\theta\to 0} \left(\frac{\sin^2\frac{\theta}{2}}{\left(\frac{\theta}{2}\right)^2} \cdot \frac{\theta}{2}\right)$

$= 1 \times 0 = 0$.

- Continuity

Def. We say $f(x)$ is continuous at c if $\lim_{x\to c} f(x)$ exists & $= f(c)$.

e.g.1 $f(x) = |x|$ contin at any c.

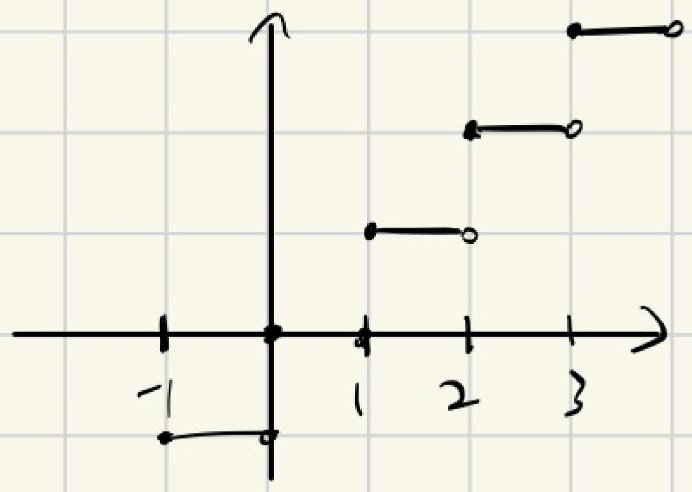
e.g.2 $f(x) =$ polynomial $p(x)$.
Recall $\lim_{x\to c} p(x) = p(c) \Rightarrow p(x)$ is contin at any c.

e.g.3 $f(x) = \frac{p(x)}{q(x)}$ > polynomials

Recall $\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$ if $q(c) \neq 0$.

e.g.4 $\sin x$ & $\cos x$ contin at any c.

e.g.5 $f(x) = [x]$. (largest integer $\leq x$)



$\lim_{x\to 1^+} f(x) = 1 \neq \lim_{x\to 1^-} f(x) = 0$

$\Rightarrow \lim_{x\to 1} f(x)$ DNE.

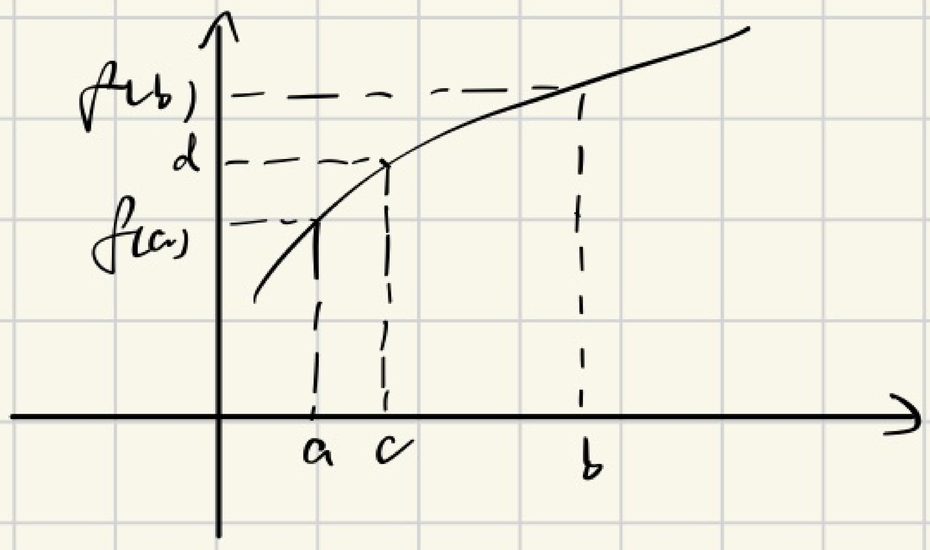
$\Rightarrow$ ($f(x)$ discontin at every c = integer.
but $f(x)$ contin at every $c \neq$ integer.)

Def. If $\lim_{x\to c^+} f(x) = f(c)$, we say $f(x)$ is right-contin at c. (same to left).

Th 1. Suppose f & g contin at c.
then so are the following at c:
① $f+g$
② $k\cdot f$ (k-const)
③ $f\cdot g$.
④ $\frac{f}{g}$ if $g(c) \neq 0$.
⑤ f^{n} / $f^{\frac{1}{n}}$. (n → pos integer)

Th 2. If f is contin at c, g contin at $f(c)$,
then $g(f(x))$ is contin at c.

Intermediate Value Theorem



Let $f(x)$ be contin on $[a,b]$.
$\forall$ d between $f(a)$ & $f(b)$.
then there exist $c \in [a,b]$.
such that $f(c) = d$.

Types of discontinuity:

① $\lim_{x\to c} f(x)$ exists, but $f(c)$ is undefined or $\lim_{x\to c} f(x) \neq f(c)$,
we say f has a removable discontinuity.

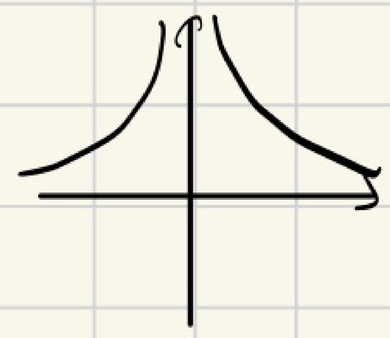
We can redefine f s.t. $f(c) = \lim_{x\to c} f(c)$,
the newly defined function is called continuous extension f at C.

② $\lim_{x\to c^+} f(x)$ & $\lim_{x\to c^-}$ exist, but $\neq$
we say f has jump discontinuity at c.

③ $f(x)$ osillates infinity with amplitude $\geq$ fixed positively #
we say f has osillating continuity at C.

e.g. $y = f(x) = \sin(\frac{1}{x})$, as $x \to 0$, $\lim_{x\to 0} \sin(\frac{1}{x})$ DNE

④ functions like $y = \frac{1}{x^2}$, as $x \to 0$, y blows up at 0.
we say f has infinite discontinuity.



Continuous function

Def. We say $f(x)$ is a continous function if:
① $f(x)$ is continuous at every interior point in its domain
② $f(x)$ is one-sided continuous at every boundary of its domain.

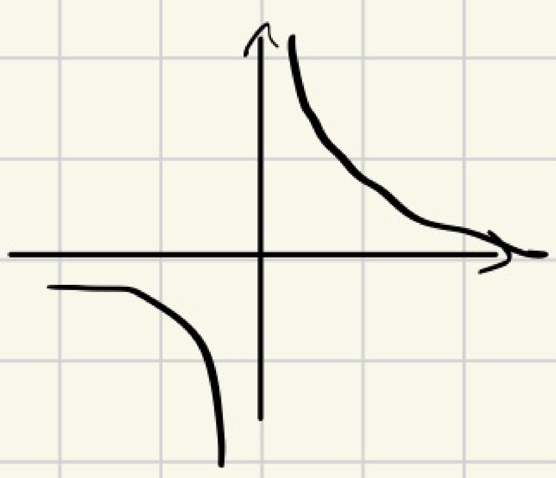
Def. We say $f(x)$ is continuous on (a,b) if f is contin at every point $\in (a,b)$.

We say $f(x)$ is continuous on $[a,b]$ if:
① $f(x)$ is contin. on (a,b).
② $f(x)$ is contin from the left at b; contin from right at a.

Limits involving infinity

Notation: $+\infty$ (positive infinity) means a veriable (independant variable x or dependant variable y) grows unstoppably large, i.e. bigger than any fixed positive #. Write $x \to +\infty$, or $y \to +\infty$ ($-\infty$ similarly)

e.g. $y = f(x) = \frac{1}{x}$
as $x \to 0^+$, $y \to +\infty$
as $x \to 0^-$, $y \to -\infty$



Def. We say limit $f(x)$ is a # L as $x \to +\infty$ if:

① f is defined on $[a, +\infty)$

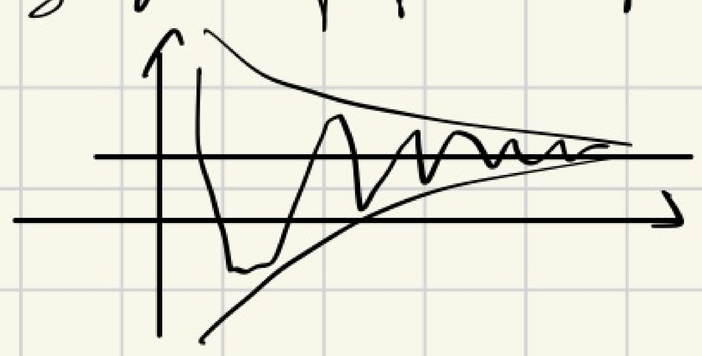
② $f(x) \to L$ as $x \to +\infty$, i.e. $f(x)$ is arbitrarily close to L - as close as we wish, whenever x is large enough.

write $\lim\limits_{x \to +\infty} f(x) = L$. ($\lim\limits_{x \to -\infty} f(x) = L$ similarly)

· Asymptotes of graphs

Def. A line $y = b$ is a horizontal asymptote of the graph of $f(x)$ if either

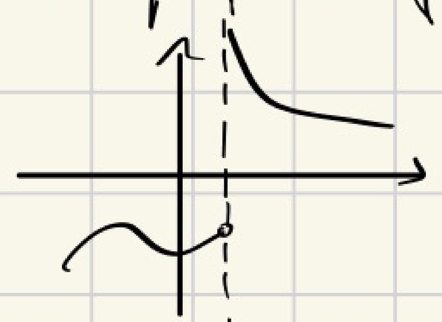
$\lim\limits_{x \to +\infty} f(x) = b$ or $\lim\limits_{x \to -\infty} f(x) = b$. e.g.



$y = b$.

Def. A line $x = c$ is a vertical asymptote of the graph of $f(x)$ of either

$\lim\limits_{x \to c^+} f(x) = \pm\infty$ or $\lim\limits_{x \to c^-} f(x) = \pm\infty$ e.g.



$x = c$

Oblique asymptotes

e.g. $y = f(x) = \dfrac{x^2+1}{x-1}$ $\Rightarrow$ $\lim\limits_{x \to 1^+} = \dfrac{2}{0^+} = +\infty$, $\lim\limits_{x \to 1^-} = \dfrac{2}{0^-} = -\infty$ $\Rightarrow$ vertical asymptote $x = 1$.

$\lim\limits_{x \to \pm\infty} f(x) = +\infty$ $\Rightarrow$ No horizontal asymptote exist

Q: No more asymptotes?

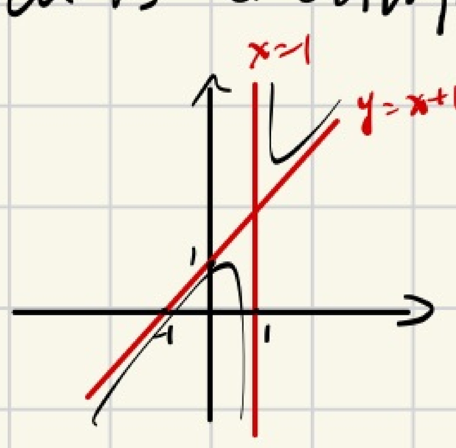
A: No! We devide (x^2+1) into $(x-1)$ $\Rightarrow$

$$\begin{array}{r} x+1 \\ x-1 \overline{\big) x^2+0\cdot x+1} \\ \underline{x^2 - x} \\ x+1 \\ \underline{x-1} \\ 2 \end{array}$$

$\Rightarrow$ $f(x) = \underbrace{(x+1)}_{\text{linear } f(x)} + \underbrace{\dfrac{2}{x-1}}_{\text{remainder}}$.

Then we have a slanted line $y = x+1$. which is a oblique asymptote of the graph of $f(x)$.

$\Rightarrow$ and we can draw the graph like



II. Derivatives

- Derivative at a point

Recall: $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ — slope of graph of $y=f(x)$ at $x=c$.
— instantaneous rate of change of f at c.

$= \lim_{x\to c}\frac{f(x)-f(c)}{x-h}$

$= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ = derivative of f at C.

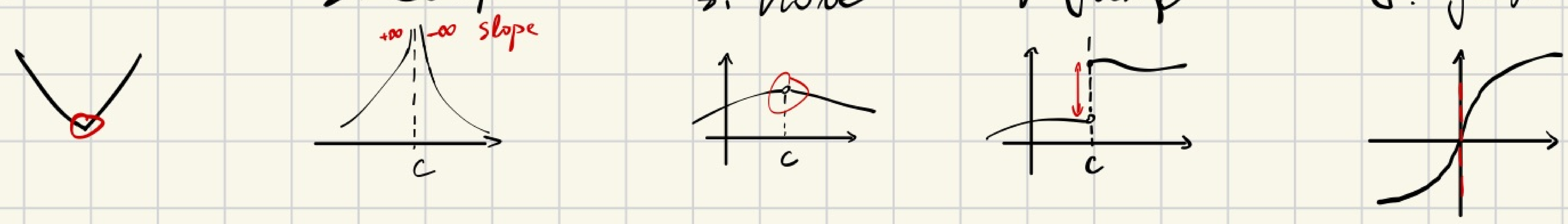
Notations: $f'(c)$, $\frac{dy}{dx}\Big|_{x=c}$, $\frac{df}{dx}\Big|_{x=c}$, $D_x f(c)$, $\dot{f}(c)$.

- Differentiability at a point

Def. We say f is differentiable at C if $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists (as a finite #)

Failure of differentiability:

1. corner
2. cusp
3. hole
4. jump
5. $y=\sqrt[3]{x}$



vertical tangent line

Theorem: differentiability $\Rightarrow$ continuity ($\not\Leftarrow$) $\lim_{x\to c} f(x)=f(c)$

Prof. Suppose f is diff. at $x=c$

$\Rightarrow f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$

we have $\lim_{x\to c}[f(x)-f(c)] = \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right]\cdot(x-c) = \lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right)\cdot\lim_{x\to c}(x-c)$

$= f'(c)\cdot 0 = 0$

$\Rightarrow \lim_{x\to c}f(x) = \lim_{x\to c}[(f(x)-f(c))+f(c)] = 0+f(c)=f(c)$.

- Derivative as a function

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{w\to x}\frac{f(w)-f(x)}{w-x}$

e.g. $f(x)=|x|$.

$\Rightarrow f'(x)=\begin{cases}1, & x>0\\ -1, & x<0\\ \text{undefined}, & x=0\end{cases}$ Domain of $f' = (-\infty,+\infty)\setminus\{0\}$

- One-sided derivatives

right-hand derivative at C

$= \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$. (left similarly)

$\Rightarrow$ right differentiable at C.

Differentiability on an interval

Def. We say f is differentiable on (a,b) if f is diff. at every $c \in (a,b)$.

We say f is diff. on $[a,b]$ if:
1. f diff. on (a,b)
2. $f'(a+0)$ & $f'(b+0)$ exist.

Theorem. $f'(c)$ exists $\iff$ $f'(c+0)$ & $f'(c-0)$ exist & equal

Differentiation Rules

1. $\frac{dc}{dx} = 0$ $(c-\text{const})$

2. $\frac{dx^n}{dx} = nx^{n-1}$ (n - postive integer)

Prof. $(x^n)' = \lim_{w \to x} \frac{w^n - x^n}{w-x}$

$= \lim_{w \to x} \frac{\cancel{(w-x)}(w^{n-1}+w^{n-2}x+w^{n-3}x^2+\dots+wx^{n-2}+x^{n-1})}{\cancel{w-x}}$

$= nx^{n-1}$

Remark: Also true is n is a $\mathbb{R}$, as long as x^n & x^{n-1} is defined.

3. $\frac{d}{dx}(f(x)+g(x)) = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$ Sum Rule

4. $\frac{d}{dx}(cf(x)) = c \cdot \frac{df(x)}{dx}$ Const Multiple Rule

$\frac{d}{dx}(c_1 f(x) + c_2 g(x))$

$= c_1 \frac{df(x)}{dx} + c_2 \frac{dg(x)}{dx}$

5. $\frac{d}{dx}(f(x) \cdot g(x)) = \frac{df(x)}{dx} g(x) + f(x) \cdot \frac{dg(x)}{dx}$ Product Rule

Prof. $(f(x)g(x))' = \lim_{w \to x} \frac{f(w)g(w) - f(w)g(x) + f(w)g(x) - f(x)g(x)}{w-x}$

$= \lim_{w \to x} \frac{f(w)(g(w)-g(x)) + g(x)(f(w)-f(x))}{w-x}$

$= f(x)g'(x) + f'(x)g(x)$.

6. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df(x)}{dx} \cdot g(x) - f(x) \cdot \frac{dg(x)}{dx}}{g^2(x)}$ Quotient Rule

Higher Order Derivatives

$(f'(x))' \Rightarrow f''(x)$ second order derivatives

In general, for integer $n \geq 1$, n-th order derivative

$f^{(n)}(x) = (f^{(n-1)}(x))' = \frac{d^n y}{dx^n} = D^n y$

- $f'(x)$ as a rate change

Chain Rule

Suppose $\begin{cases} g(x) \text{ is differentiable at } x=c \\ f(u) \text{ is differentiable at } u=g(c) \end{cases}$

Then $f \circ g(x)$ is diff. at $x=c$, moreover, $(f \circ g)'(c) = f'(g(c))\, g'(c)$.

$f(g(x))' = f'(g(x)) \cdot g'(x)$.

In Leibniz's notation:

$$\left.\frac{dy}{dx}\right|_{x=c} = \left.\frac{dy}{du}\right|_{u=g(c)} \cdot \left.\frac{du}{dx}\right|_{x=c}$$

Proof of Chain Rule:

$$\left.\frac{dy}{dx}\right|_{x=c} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}, \text{ where } \Delta y = f(g(c+\Delta x)) - f(g(c)).$$

$$= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}, \text{ where } \Delta u = g(c+\Delta x) - g(c).$$

$$= \lim_{\Delta x\to 0}\frac{f(g(c)+\Delta u) - f(g(c))}{\Delta u}\cdot \lim_{\Delta x\to 0}\frac{g(c+\Delta x)-g(c)}{\Delta x}.$$

($\Delta u \neq 0$!)

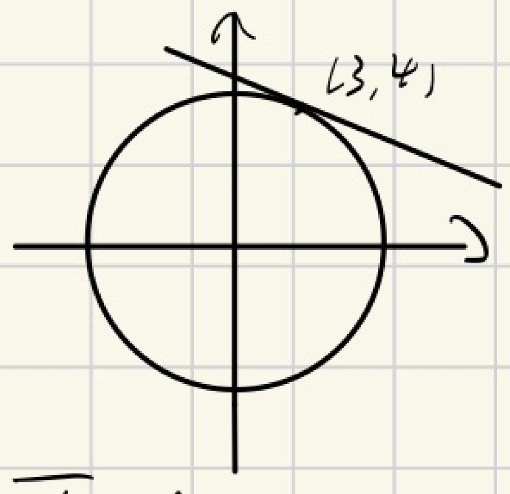
e.g. $y = (x^2+1)^{2024}$

$y' = 2024(x^2+1)^{2023}\cdot 2x$.

· **Implicit Differentiation**

e.g. $x^2+y^2=25$

$\Rightarrow y = \pm\sqrt{25-x^2}$



Want: equation of T-line passing through (3,4).

$$\text{Slope } m = \left.\frac{d\sqrt{25-x^2}}{dx}\right|_{x=3} = \left.\frac{d(25-x^2)^{\frac{1}{2}}}{dx}\right|_{x=3} = \left.\frac{1}{2}(25-x^2)^{-\frac{1}{2}}\cdot(-2x)\right|_{x=3}$$

$$= -\frac{3}{4} \quad \text{(complex way).}$$

▷ Better way:

$$\frac{d}{dx}(x^2+y^2=25). \quad \Rightarrow \quad \frac{dy^2}{dx} = \frac{dy^2}{dy}\cdot\frac{dy}{dx} = 2y\cdot\frac{dy}{dx}.$$

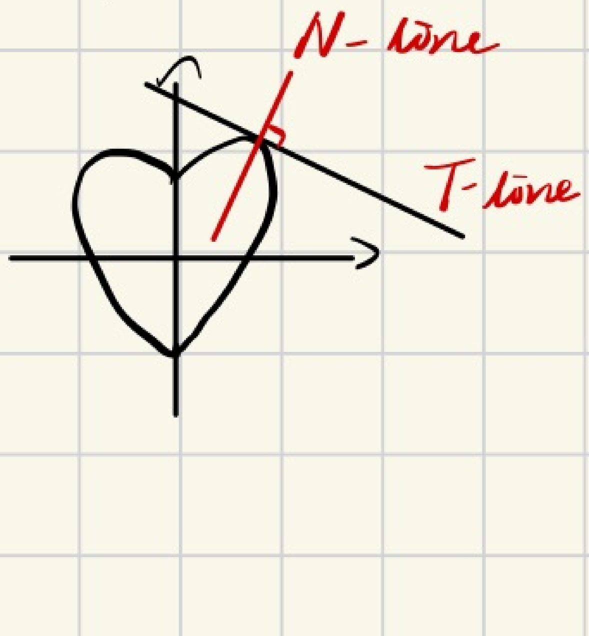
Chain Rule

$$2x + 2y\cdot\frac{dy}{dx} = 0.$$

$$\Rightarrow m = \left.\frac{dy}{dx}\right|_{(x,y)=(3,4)} = \left.-\frac{x}{y}\right|_{(x,y)=(3,4)} = -\frac{3}{4}. \quad \text{(Smart way)}$$

· Normal lines

Def. Lines that are $\perp$ T-line, passing through the point of tangency.



Case 1: Slope of T-line $m \neq 0, \pm\infty$. $(k_1\cdot k_2 = -1)$.

$\Rightarrow$ Slope of N-line $= -\frac{1}{m}$.

Case 2: Slope of T-line $m=0$.

$\Rightarrow$ N-line is vertical $\Rightarrow$ equation of N-line is $x=a$.

Case 3: Slope of T-line $m = \pm\infty \Leftrightarrow$ T-line vertical

$\Rightarrow$ N-line' eq: $y=b$.

Related Rates

Scenario: Have physical, biological, financial problem involving independant variable t, and dependant variables $x, y, u, v \dots$ which are related by equation. (e.g. $x^2 + \sin(yuv) + t^2 = 9.6$) Suppose we are given $\frac{dy}{dt}$, $\frac{du}{dt}$, $\frac{dv}{dt}$. And we want: $\frac{dy}{dx}$.

Recipe:
1. Draw a diagram & label important varibles.
2. Write what's given in mathematical terms.
3. Write what we want in M terms.
4. Derive eq relating the variables
5. Diff the eq w.r.t. the independant variable.

Linearization and Differentials

Recall: if f is differential at $x=a$, then

$$f'(a) \text{ exist } \& = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

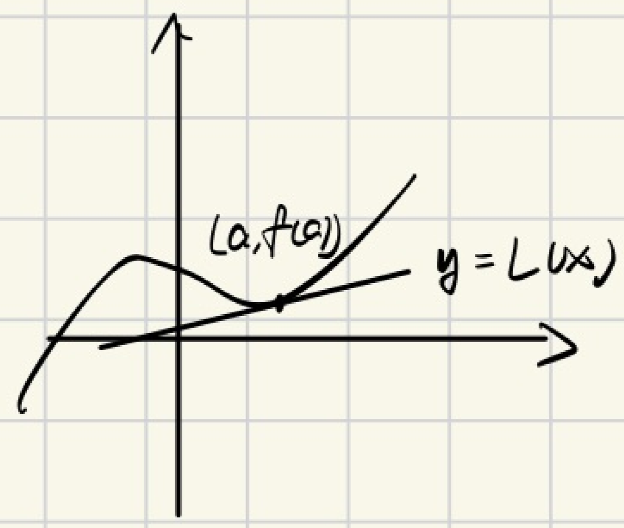
$$\Rightarrow \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a} - f'(a)\right] = 0$$

$$\Rightarrow \frac{f(x)-f(a)}{x-a} - f'(a) \approx 0 \text{ if } x \approx a, \neq a.$$

$$\Rightarrow f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{L(x)} \text{ if } x \approx a.$$

Linearization of $f(x)$ at $x=a$.

$$y = L(x) \Leftrightarrow y - f(a) = f'(a)(x-a).$$



e.g. $f(x) = \tan x$. $f'(x) = \sec^2 x$

$L(x) = f(0) + f'(0)(x-0) = 0 + 1 \cdot x = x$.

(T-line at origin)

e.g. $f(x) = (1+x)^k$, $k \in \mathbb{R}$ $\quad f'(x) = k(1+x)^{k-1}$

T-line at origin

$L(x) = f(0) + f'(0)\cdot x = 1 + kx$

$\Rightarrow f(x) \approx 1 + kx$ when $x \approx 0$. Application of Linearization.

Approximation

sp. 1. $k = \frac{1}{2}$, $\sqrt{1+x} \approx 1 + \frac{1}{2}x$. $(x \approx 0)$.

2. $k = -1$. $\frac{1}{1+x} \approx 1 - x$ $(x \approx 0)$.

3. $k = -\frac{1}{2}$ $\frac{1}{\sqrt{1+x}} \approx 1 - \frac{x}{2}$ $(x \approx 0)$.

$$f(x) \approx L(x) = f(a) + f'(a)(x-a), \quad x \approx a.$$

Let $h = x - a \Rightarrow x = a + h$

$$f(a+h) = f(a) + f'(a)h, \quad h \approx 0.$$

e.g. $\sqrt{9.01} = ?$ We have $\sqrt{9} = 3$.

Let $f(x) = \sqrt{x}$. $f(9) = 3$. $f(9.01) = ?$ $\Rightarrow f(9.01) = f(9 + 0.01) = f(9) + f'(9)\cdot 0.01$

$= 3 + \frac{1}{6}\cdot 0.01$

$= 3.001666\dots$

error $= 4.627\dots \times 10^{-7}$ Very accurate approximation!

Q: How good is the approximation?

A: Recall $\lim\limits_{x\to a}\left[\frac{f(x)-f(a)}{x-a}-f'(a)\right]=0$

Let $\varepsilon = \frac{f(x)-f(a)}{x-a}-f'(a)$. we have $\lim\limits_{x\to a}\varepsilon=0$.

$\varepsilon(x-a) = f(x)-f(a)-f'(a)(x-a)$

$\Rightarrow f(x) = \overbrace{f(a)+f'(a)(x-a)}^{L(x)} + \varepsilon(x-a)$. ($\varepsilon\to 0$ as $x\to a$).

error $= |f(x)-L(x)| = |\varepsilon|\cdot|x-a|$.

$\Rightarrow \frac{\text{error}}{x-a}\to 0$ as $x\to a$. $\Rightarrow$ error is extremely tiny!!

small "o".

Notation: ① $o(1)$ represents a function which $\to 0$ as $x\to a$ (a may $\pm\infty$).

e.g. $\sin x = o(1)$ as $x\to 0$.
$\sin x = o(1)$ as $x\to\pi$
$x^a = o(1)$ as $x\to 0$.

② $o(g(x)) \overset{\text{defined}}{=\!=} g(x)\,o(1)$.

e.g. $o(x^{423}+516x^{215}+2024) = (x^{423}+516x^{215}+2024)\cdot o(1)$.

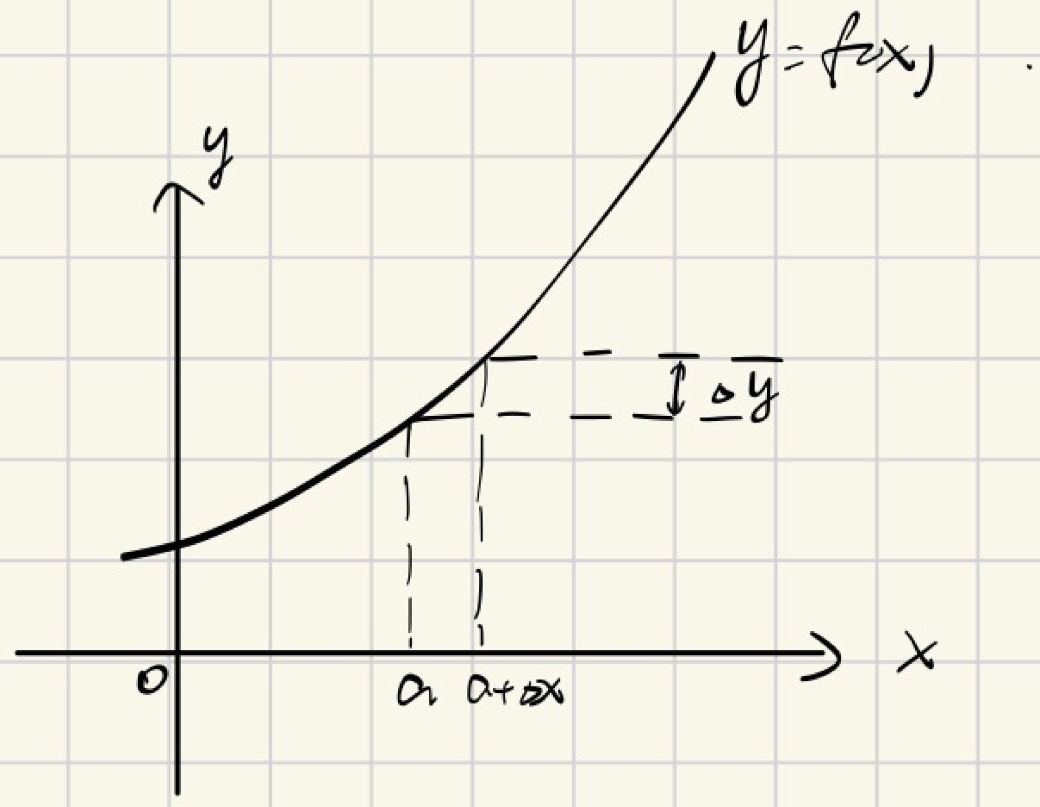
big "O"

③ $O(1)$ represents a function which is bounded as $x\to a$.

$\exists M>0$ s.t. $|f(x)|\le M$, $x\approx a$.

e.g. $\sin x = O(1)$ as $x\to\infty$.

Differential

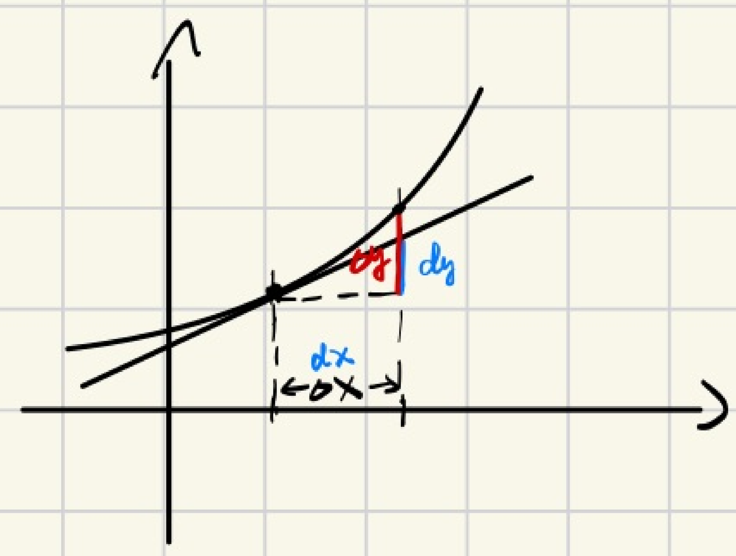


Motivation: Suppose f is diff at $x=a$.

$\Delta y = f(a+\Delta x)-f(a)$
$= f(a)+f'(a)\Delta x+o(\Delta x)-f(a)$.
$= f'(a)\Delta x + o(\Delta x)$ if $\Delta x\approx 0$.

$\Delta y \approx f'(a)\Delta x \iff \frac{\Delta y}{\Delta x}\approx f'(a)$.

$\left.\frac{dy}{dx}\right|_{x=a} = \lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$.



write $dx = \Delta x$. we have $\Delta y\approx f'(a)dx$

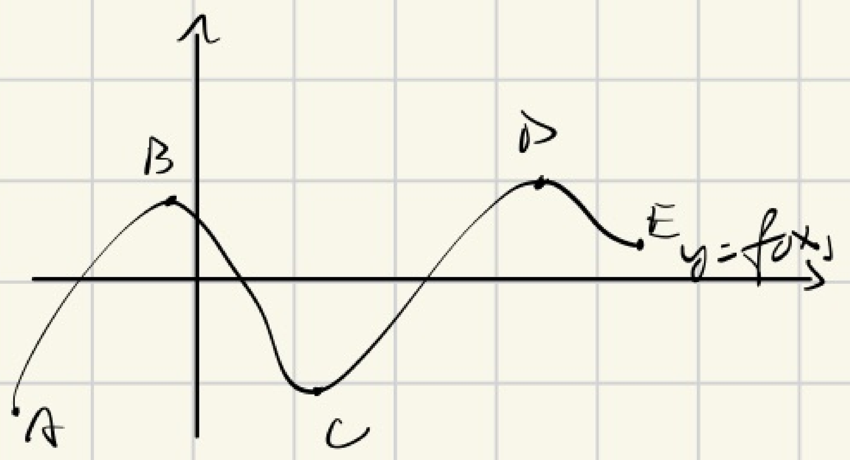
Def. $f'(a)dx$ is called differential of f at a.

Notation: $\left.dy\right|_{x=a} = f'(a)dx$.

Thus $\Delta y \approx \left.dy\right|_{x=a}$ differential

difference/changing.

Extreme values of functions.



Local min: A, C, E.
Local max: B, D.
Absolute/global min: A.
Absolute/global max: D.

△ Extreme value Theorem: Suppose $f(x)$ contin. on $[a,b]$.
Then f has abs. max & min attained in $[a,b]$.

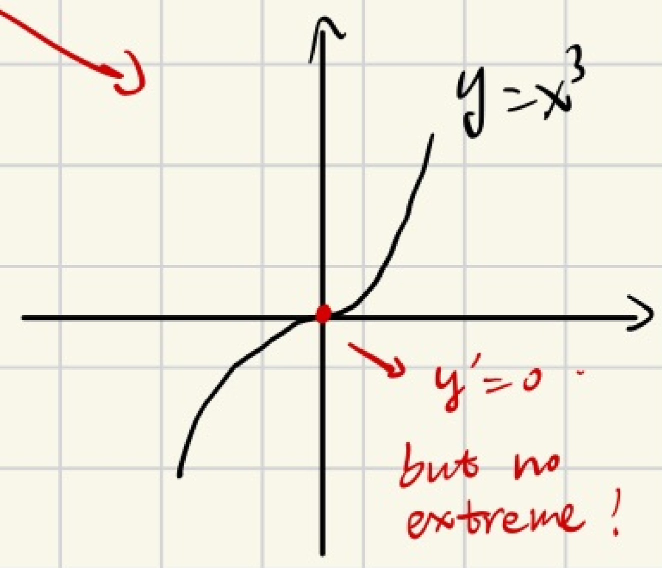
Q: How to find extrema?

△ Theorem: First Derivative theorem for local extrema:
If $f(c)$ is local extrema & $f'(c)$ exists, then $f'(c)=0$

$x=c$: Critical Point

Corollary: Local extrema can occur only at points where either
$f'(c)=0$ / $f'(c)$ DNE / at endpoints of domain.

Warning: If $f'(c)=0$, then $f(c)$ may or may not be the extreme value.

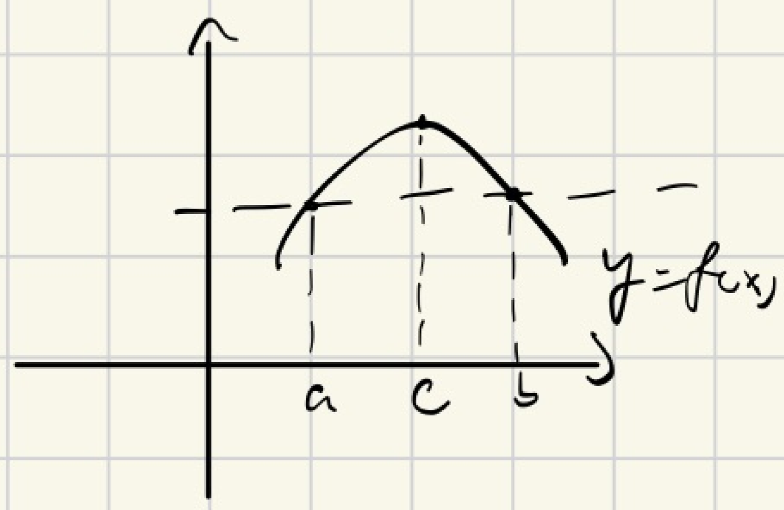


Recipe to find abs. max & abs min of f (contin) on $[a,b]$.

Step 1. Find all c.p. of f in (a,b).
Step 2. Evaluate f at these c.p.'s & $f(a)$ & $f(b)$.
Step 3. Choose the largest function value $\rightarrow$ abs. max
- - - - smallest — — . $\rightarrow$ abs. min

Mean Value Theorem

Baby case of MVT: Rolle's Theorem

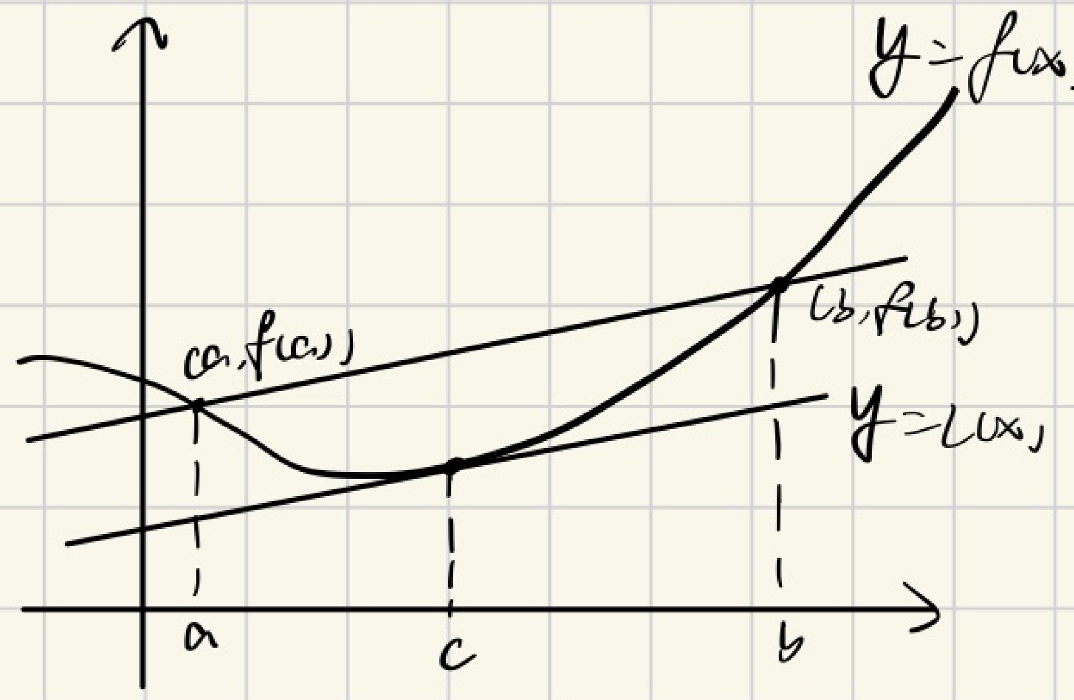


If $f(x)$ is contin on $[a,b]$, and differentiable on (a,b).
Assume $f(a)=f(b)$, then $\exists\, c\in(a,b)$ s.t. $f'(c)=0$

Proof: Since f is contin on $[a,b]$, by "Extreme Value Th",
$\exists\, x_{min}$ & $x_{max}\in[a,b]$ s.t. $f(x_{min})=$ abs. min, $f(x_{max})=$ abs. max.

Then we have:
Case 1: Either $x_{min}\in(a,b)$ or $x_{max}\in(a,b)$.
by First Derivative Th., take $c=x_{min}/x_{max}$, then $f'(c)=0$.
Case 2: $f(x_{max})=f(x_{min})=f(a)=f(b)$.
$\Rightarrow f(x)\equiv$ const, take $c=$ any point, then $f'(c)=0$. QED

General Mean Value Theorem



Suppose f contin on $[a,b]$, diff on (a,b).
Then $\exists\, c \in (a,b)$ s.t.

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c)}$$

$L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$.

Proof: Let $g(x) = L(x) - f(x)$
Want to apply Rolle to $g(x)$ on $[a,b]$.
1. $g(x)$ contin on $[a,b]$.
2. $g(x)$ diff on (a,b)
3. $g(a) = g(b) = 0$

Now by Mr. Rolle, $\exists\, c \in (a,b)$ s.t. $g'(c) = 0$ $\Rightarrow L'(x) - f'(x) = 0$.
$\Rightarrow f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. QED.

Corollary 1: Suppose f' exist & $\equiv 0$ on (a,b)
$\Rightarrow f \equiv$ const on (a,b)

Corollary 2: Suppose $f'(x) = g'(x)$ on (a,b)
$\Rightarrow f(x) = g(x) +$ const C on (a,b).

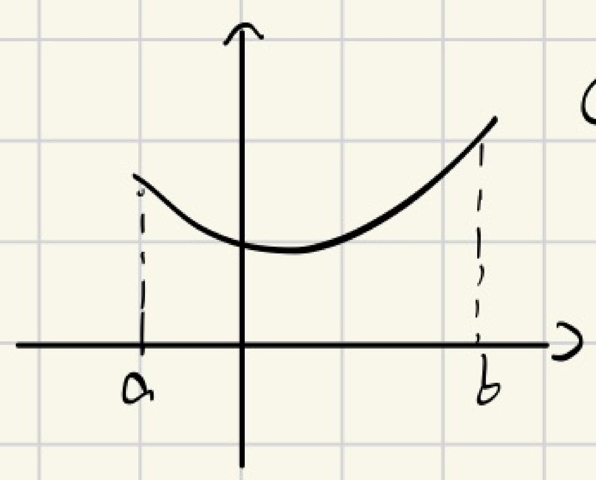
Corollary 3: Suppose $f'(x) > 0$ on (a,b)
$\Rightarrow f$ is increasing, i.e. $\forall\, a < x_1 < x_2 < b$, $f(x_1) < f(x_2)$.
(Similarly with $f'(x) < 0 \Rightarrow$ decreasing).

· First Derivative Test for Local Extrema

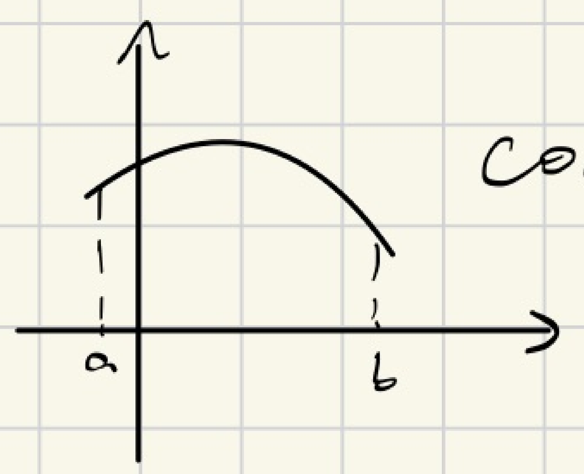
Suppose c is c.p. of f, and f is contin in nbhd of c.
In the interval containing c, moving from left to right,
1. If f' changes from negative to positive at c, then f has a local minimum at c.
2. - - - - positive to negative - - - - - - maximum - -
3. If f' doesn't change sign, then f has no local extremum at c.

· Concavity and Curve Sketching

Def. We say the graph of $y = f(x)$ is concave up on interval (a,b) if $f(x)$ is increasing on (a,b) (concave down $\to$ decreasing)



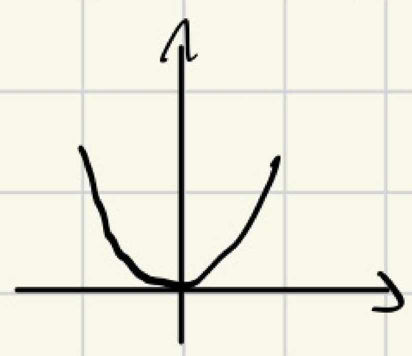
concave up



concave down

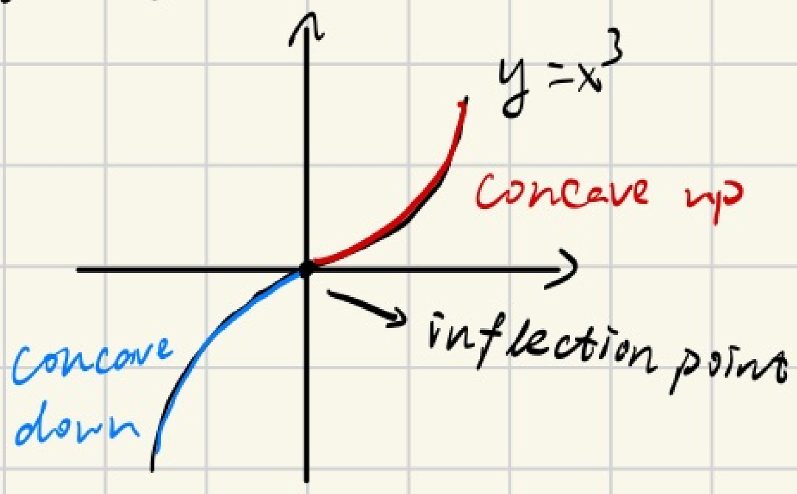
Rk. If $f''(x) > 0$ (< 0) on (a,b) then $f'(x)$ is $\uparrow$ ($\downarrow$) on (a,b) $\Rightarrow f(x)$ is concave up (down).

e.g. $f(x) = x^2$. $f'(x) = 2x$. $f''(x) = 2$.



$\Rightarrow f(x)$ is concave up.

e.g. $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x = \begin{cases} \oplus, & x > 0 \\ \ominus, & x < 0 \end{cases}$



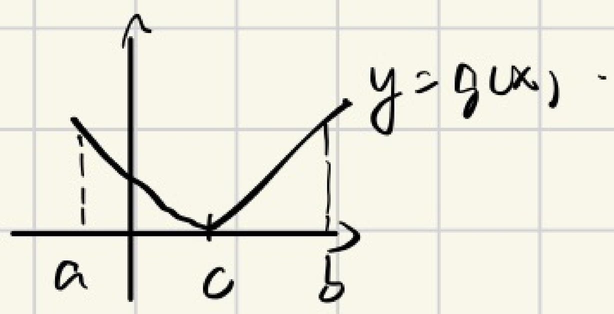
Def. A point $(c, f(c))$ on the graph $y = f(x)$ is called inflection point if the T-line at $(c, f(c))$ exists (including vertical T-line) and concavity changes at $(c, f(c))$.

Theorem 1: Suppose f is differentiable on (a,b) and $f(x)$ concave up (down) on (a,b).
Then $\forall (c, f(c))$ where $c \in (a,b)$, the graph of $f(x)$ stays above (below) the T-line passing through $(c, f(c))$, i.e.

$$f(x) > f(c) + f'(c)(x-c), \ \forall x \in (a,b) \neq c.$$

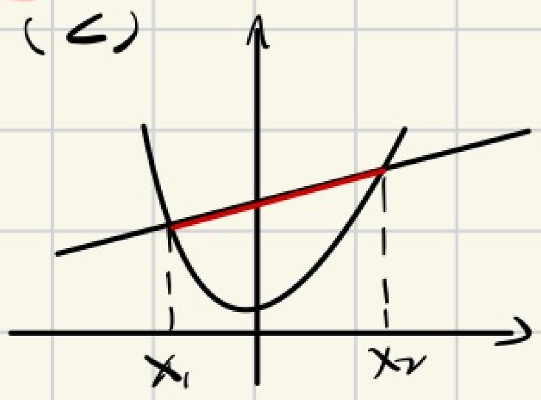
Proof. Define $g(x) = f(x) - l(x)$. want: $g(x) > 0$, $\forall x \in (a,b) \neq c$.

Observe: 1. $g(c) = 0$
2. $g(x)$ is diff on (a,b)
3. $g'(x) = f'(x) - f'(c) = \begin{cases} \oplus, & c < x < b \\ \ominus, & a < x < c \end{cases}$



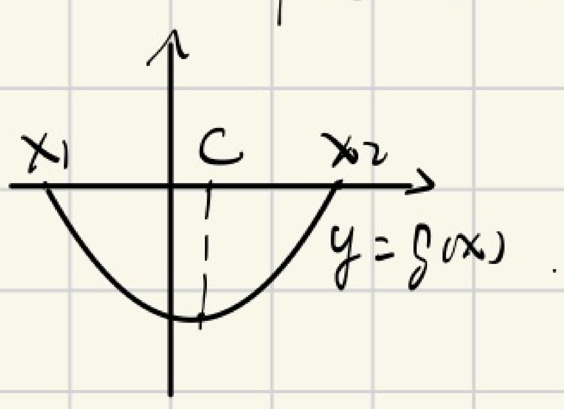
$\Rightarrow g(x) > 0$ on $(a,c) \cup (c,b)$.
i.e. $f(x) > l(x)$.

Theorem 2: Suppose f is diff on (a,b) & is concave up (down) on (a,b).
Then $\forall a < x_1 < x_2 < b$, $\underbrace{f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)}_{l(x)} > (<) f(x)$, $x \in (x_1, x_2)$.



Proof. Define $g(x) = f(x) - l(x)$
$\Rightarrow$ 1. $g(x_1) = 0 = g(x_2)$
2. g diff on $[x_1, x_2]$.
3. $g'(x) = f'(x) - f'(c)$*
$= \begin{cases} \oplus, & c < x < x_2 \\ \ominus, & x_1 < x < c. \end{cases}$

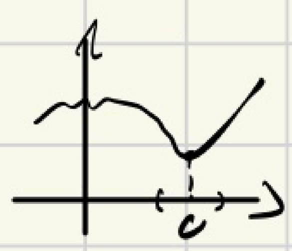
*By MVT, $\exists c \in (x_1, x_2)$ s.t. $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$.

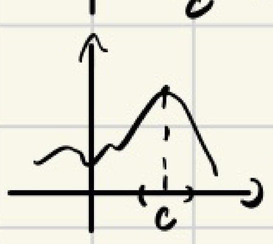


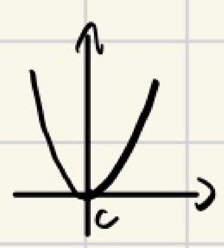
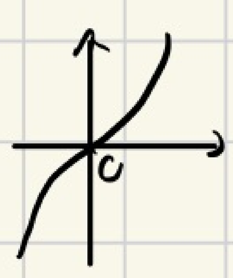
$\Rightarrow g(x) < 0$ on (x_1, x_2).
i.e. $f(x) < l(x)$.

- **Second Derivative Test** for Local Extrema.

Suppose f'' exists & contin on (a,b) & $c \in (a,b)$ is critical point ($f'(c) = 0$).

Case 1: $f''(c) > 0$  $\Rightarrow$ local min at $x = c$

Case 2: $f''(c) < 0$  $\Rightarrow$ local max at $x = c$.

Case 3: $f''(c) = 0$  or  $\Rightarrow$ Inconclusive!

e.g. $y = 2\cos x - \sqrt{2}x$, $x \in [-\pi, \frac{3}{2}\pi]$. loc extrema? inflection pts?

Step 1. Compute y', y''.

$y' = -2\sin x - \sqrt{2}$

$y'' = -2\cos x$

Step 2. Find c.p.s

Sets $y' = 0 \Rightarrow x = -\frac{\pi}{4}, \frac{5\pi}{4}, -\frac{3\pi}{4}$. (loc extreme)

Step 3. Set $y'' = 0 \Rightarrow x = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}$ (inflection pts).

- Applied Optimization

Step 1. Read the problem

Step 2. Draw diagram & label important variables / quantities

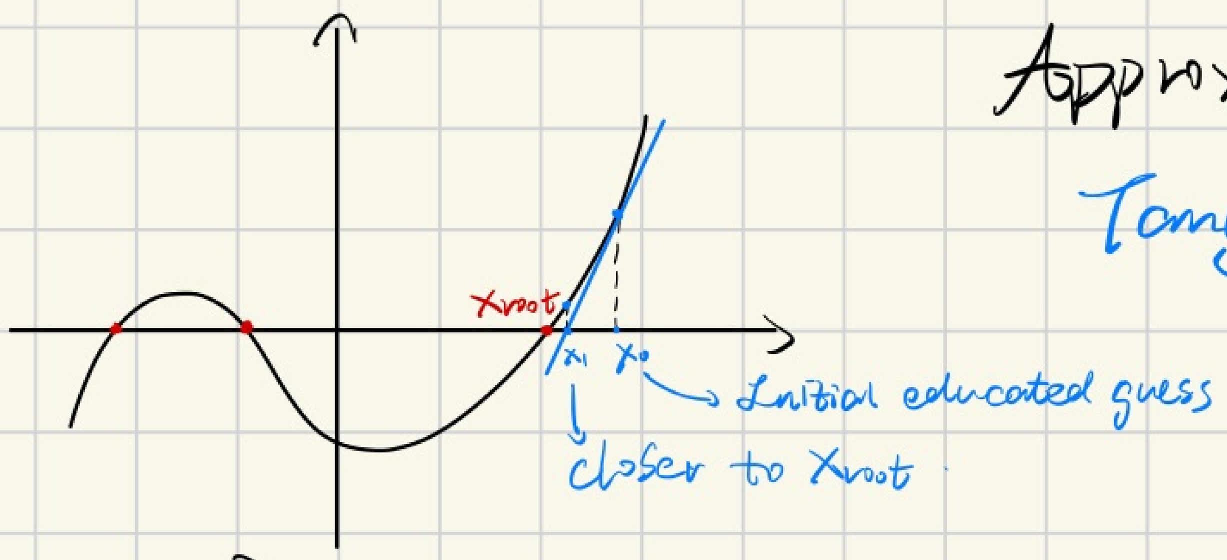
Step 3. Write down what you are given & want to know

Step 4. Do math

- Newton's method

Goal: Solve $f(x) = 0$.

Idea: sols = intersections of graph with x-axis.



Approximate Xroot:

Tangent line!

$x_1 = ?$

T-line passing through $(x_0, f(x_0))$:

$L(x): y = f(x_0) + f'(x_0)(x - x_0)$

Set $L(x) = 0$. We have:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Then. $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

$\vdots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in \mathbb{N}^*.$$

IV. Integrals

· Antiderivatives (Indefinite Integrals)

Central Q: Given $f(x)$ defined on interval I, find all the functions $F(x)$ s.t. $F'(x) = f(x)$.

e.g. $f(x) = x^a$.

$F(x) = \frac{x^{a+1}}{a+1} + C$. $\quad (F'(x) = f(x))$

Theorem: If $F(x)$ is a particular antiderivative of $f(x)$, then any other antiderivative $G(x)$ of $f(x)$ is given by $G(x) = F(x) + C$. (C is a const).

Proof: Since both $F(x)$ & $G(x)$ are antiderivatives of $f(x)$ on I, we have:

$F'(x) = G'(x)$, $\forall x \in I$

$[F(x) - G(x)]' = 0$, $\forall x \in I$

$\Rightarrow F(x) - G(x) = C$. Q.E.D.

Def: The collection of all derivatives of $f(x)$ on I is called indefinite integral of $f(x)$ on I.

Notation: $\boxed{\int f(x)dx = F(x) + C}$

e.g. $\int x^a dx = \frac{x^{a+1}}{a+1} + C$.

★ Rules for $\int f(x)dx$:

1. $\int \tilde{c}\, dx = \tilde{c}x + C$.

2. $\int (c_1 f(x) + c_2 g(x))\, dx = c_1 F(x) + c_2 G(x) + C$

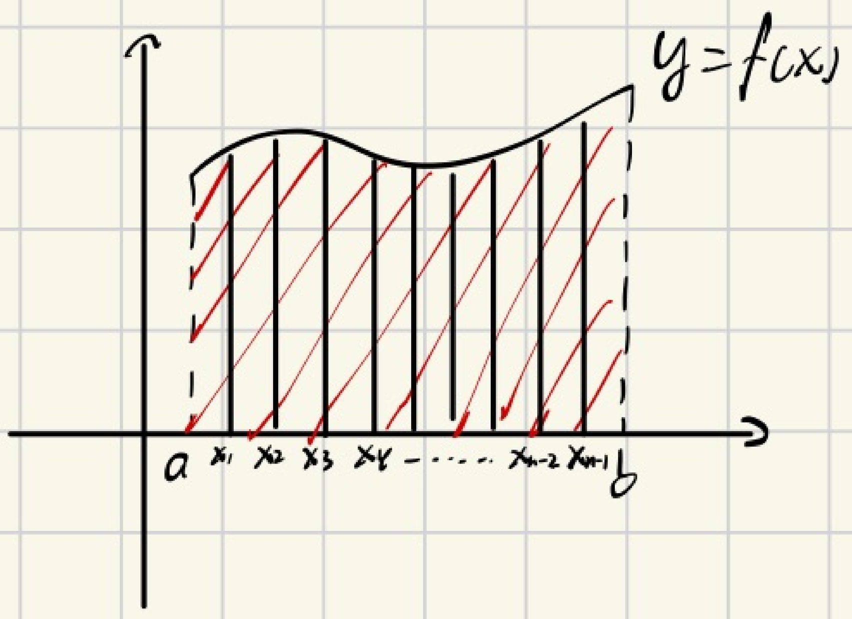
$= c_1 \int f(x)\, dx + c_2 \int g(x)\, dx$ Sum Rule

3. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ (Initial Value Problem).

$\Rightarrow \int \frac{d^2y}{dx^2}\, dx = \frac{dy}{dx} + C$.

· Definite Integrals

Motivation 1: Let $y = f(x)$ be defined on $[a,b]$, $f(x) \geq 0$, $\forall x \in [a,b]$.



Want: Area of $f(x)$ below graph & above interval $[a,b]$

Start with "Partition of $[a,b]$"

$P = \{x_0, x_1, x_2, \cdots x_{n-1}, x_n\}$. $(x_1 = a,\ x_n = b)$

In $[x_i, x_{i+1}]$, pick $c_i \in [x_i, x_{i+1}]$, use $f(c_i)$ as height

Then $S_i \approx f(c_i)\underbrace{(x_{i+1} - x_i)}_{\Delta x_i}$.

★ $S = \sum_{i=1}^{n} S_i \approx \sum_{i=1}^{n} f(c_i)\Delta x_i$ when $\|P\|^*$ is tiny — Riemann Sum.

* $\|P\|$: Norm of Partition P. $= \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\}$

Expectation:

$\sum_{i=1}^{n} f(c_i)\Delta x_i \to S$ as $\|P\| \to 0$.

Def: If $\lim\limits_{\|P\|\to 0}\sum\limits_{i=1}^{n} f(c_i)\Delta x_i$ exist & is independent of choice of c_i's,

then we say it is the definite integral of f on $[a,b]$.

Notation: $\boxed{\int_a^b f(x)\,dx = \lim\limits_{\|P\|\to 0}\sum\limits_{i=1}^{n} f(c_i)\Delta x_i}$

Def: Area S should be defined as $\int_a^b f(x)\,dx$ if $f(x)\geq 0$ on $[a,b]$.

Def: If always chose c_i's to be left end point (right end / mid), i.e. $c_i = x_i$ (x_{i+1} / $\frac{x_i+x_{i+1}}{2}$) $(i=0,1,\cdots,n-1,n)$

then $\sum\limits_{i=1}^{n} f(c_i)\Delta x_i$ is called left (right / mid) sum.

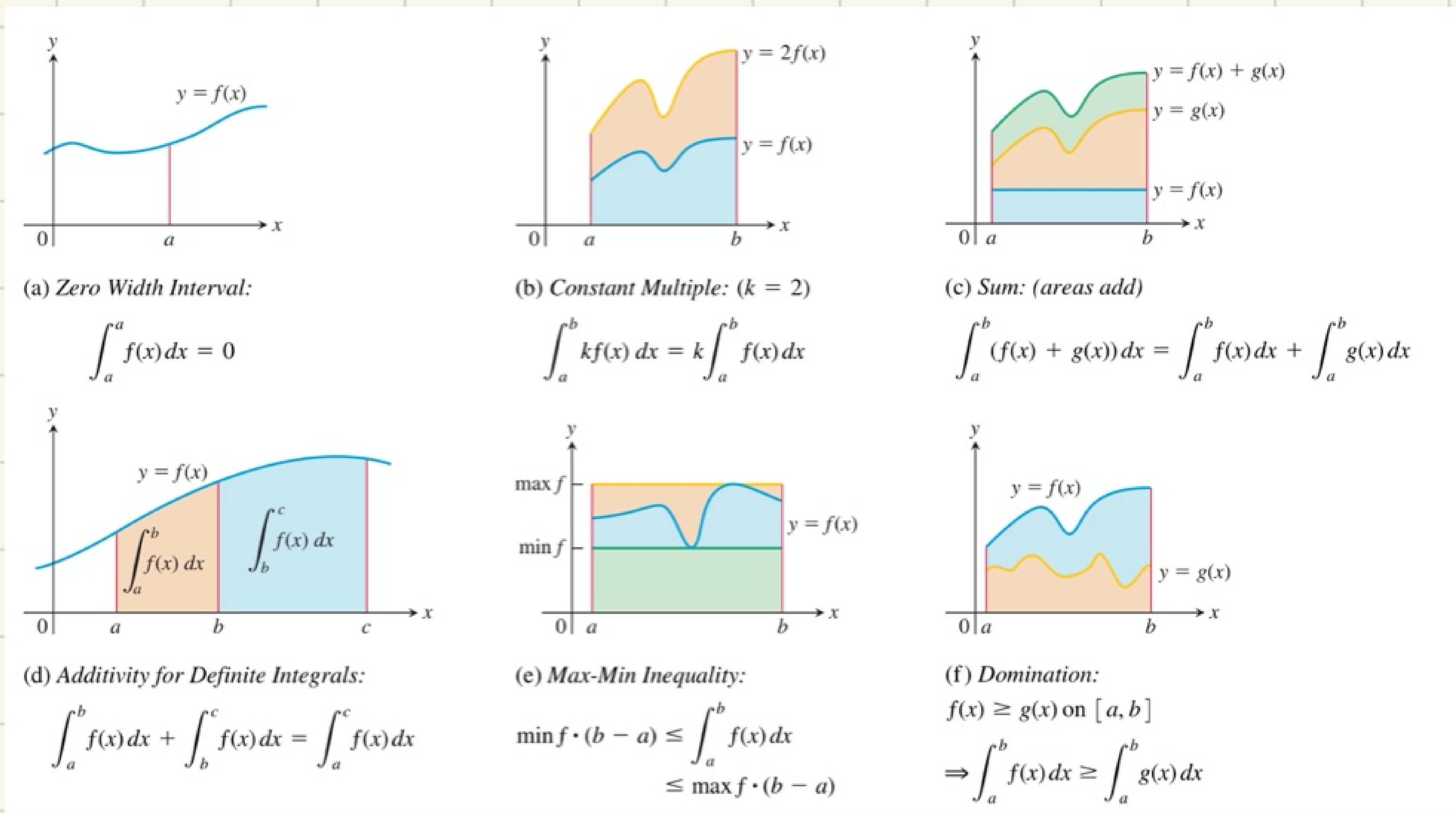
Theorem 1: Integrability of Continuous Functions

① f is continuous over the interval $[a,b]$.

or ② f has at most finitely many jump discontinuities on $[a,b]$.

$\Rightarrow$ We say $\int_a^b f(x)\,dx$ exists & f is integrable over $[a,b]$.

e.g. $f(x)=\begin{cases}1, & x\in R\\ 0, & x\notin R\end{cases}$. $\Rightarrow$ $f(x)$ has no integrability.



(a) *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$

(b) *Constant Multiple:* $(k = 2)$ $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

(c) *Sum: (areas add)* $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(d) *Additivity for Definite Integrals:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(e) *Max-Min Inequality:* $\min f\cdot(b-a) \leq \int_a^b f(x)\,dx \leq \max f\cdot(b-a)$

(f) *Domination:* $f(x)\geq g(x)$ on $[a,b]$ $\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

★ Definite Integral Rules

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$. Zero Width

3. $\int_a^b kf(x)\,dx = k\cdot\int_a^b f(x)\,dx$ Constant Multiple

4. $\int_a^b (f(x)\pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ Sum

5. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ Additivity

6. If f has max & min on $[a,b]$, then Max – Min Inequality.

$f(x)_{min}\cdot(b-a) \leq \int_a^b f(x)\,dx \leq f(x)_{max}\cdot(b-a)$

7. $f(x)\geq g(x)$ on $[a,b]$ $\Rightarrow$ $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

· Average Value of Con. f.

Def. If f is integrable on $[a,b]$, then its average value (mean) on $[a,b]$ is:

$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

- **Mean Value Theorem** for Definite Integrals

Def. If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$\triangle \quad f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Proof. By Min-Max inequality, we have

$$\min f \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \max f.$$

By the Intermediate Value Theorem, in the con. f, there exist $f(c) \in [\min f, \max f]$

Thus, $\exists c \in [a,b]$, $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

- **The Fundamental Theorem of Calculus** (Part I).

Suppose f is continuous on $[a,b]$. Let $F(x): [a,b]$, $F(x) = \int_a^x f(t)\,dt$,

then: ① $F(x)$ is continuous on $[a,b]$,

② $F(x)$ is differentiable on (a,b).

③ Its derivative is $f(x)$.

$$\star \quad F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Proof: $F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}$

By the MVT for integrals. $\exists c \in (x, x+h)$ s.t. $f(c) = \frac{\int_x^{x+h} f(t)\,dt}{h}$

As $h\to 0$, $c \to x$, s.t. $\lim_{h\to 0} f(c) = f(x)$.

Thus. $F'(x) = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h\to 0} f(c) = f(x)$. Q.E.D.

- **The Fundamental Theorem of Calculus** (Part II).

If f is continuous on $[a,b]$, and F is any antiderivative of f on $[a,b]$, then:

$$\star \quad \boxed{\int_a^b f(x) = F(b) - F(a)}$$ Newton-Leibniz formula.

Proof: FTC I tells us that an antiderivative of f exists.

Let $G(x) = \int_a^x f(t)\,dt$

Since $F(x)$ is any antiderivative of f, we have:

$F(x) = G(x) + C$.

Thus, $F(b) - F(a) = (G(b)+C) - (G(a)+C)$

$= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt$

$= \int_a^b f(t)\,dt$ Q.E.D.

Notation: $F(b) - F(a) = F(x)\Big]_a^b = \Big[F(x)\Big]_a^b$

e.g. $\int_0^{\pi} \cos x\,dx = \sin x\Big]_0^{\pi} = \sin\pi - \sin 0 = 0$.

· Substitution Method

Def. If $u=g(x)$ is a differentiable function whose range is an interval I, and f is contin. on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$ The Substitution Rule.

Proof. $\int f(g(x))\, g'(x)\, dx = \int f(u) \cdot \frac{du}{dx} \cdot dx = \int f(u)\, du$.

e.g. $\int \sin^3 x\, dx = \int \sin^2 x \cdot \sin x\, dx = \int \sin^2 x\, d(-\cos x) = \int (\cos^2 x - 1)\, d(\cos x)$

let $u = \cos x$. we have $\int (u^2-1)\, du = \frac{1}{3}u^3 - u + C = \frac{1}{3}(\cos^3 x) - \cos x + C$.

e.g. $\int f(Ax+B)\, dx = \int f(u)\, d(\frac{u-B}{A}) = \int f(u) \frac{1}{A}\, du = \frac{1}{A} F(u) + C = \frac{1}{A} F(Ax+B) + C$.

e.g. $\int \cos(7x+3)\, dx = \int \cos u\, d(\frac{u-3}{7}) = \frac{1}{7}\int \cos u\, du = \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7x+3) + C$.

Def. If g' is contin. on the interval $[a,b]$ and f is contin. on the range of $g(x)$, let $u = g(x)$. then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$ Substitution in Definite Integrals

Proof. $\int_a^b f(g(x))\, g'(x)\, dx = F(g(x))\Big]_a^b = F(u)\Big]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\, du$

$F(g(x))' = F'(g(x)) \cdot g'(x) = f(g(x))\, g'(x)$.

e.g. $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\, dx$, let $u = x^3+1$. then $du = 3x^2 \cdot dx$.

thus. $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\, dx = \int_0^2 \sqrt{u}\, du = \frac{2}{3}u^{\frac{3}{2}}\Big]_0^2 = \frac{4\sqrt{2}}{3}$

· Definite Integrals of Symmetric Functions

Def. Let f be contin. on the symmetric interval $[-a,a]$.

① If f is even, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$

② If f is odd, then $\int_{-a}^{a} f(x) = 0$.

P.S. Below contents are applications of integrals.

· Volumes Using Cross-Sections

Def. The volumn of a solid of integrable cross-sectional area $A(x)$ from $x=a$ to $x=b$:

$$V = \int_a^b A(x)\, dx$$

similar to evaluate area on plane.

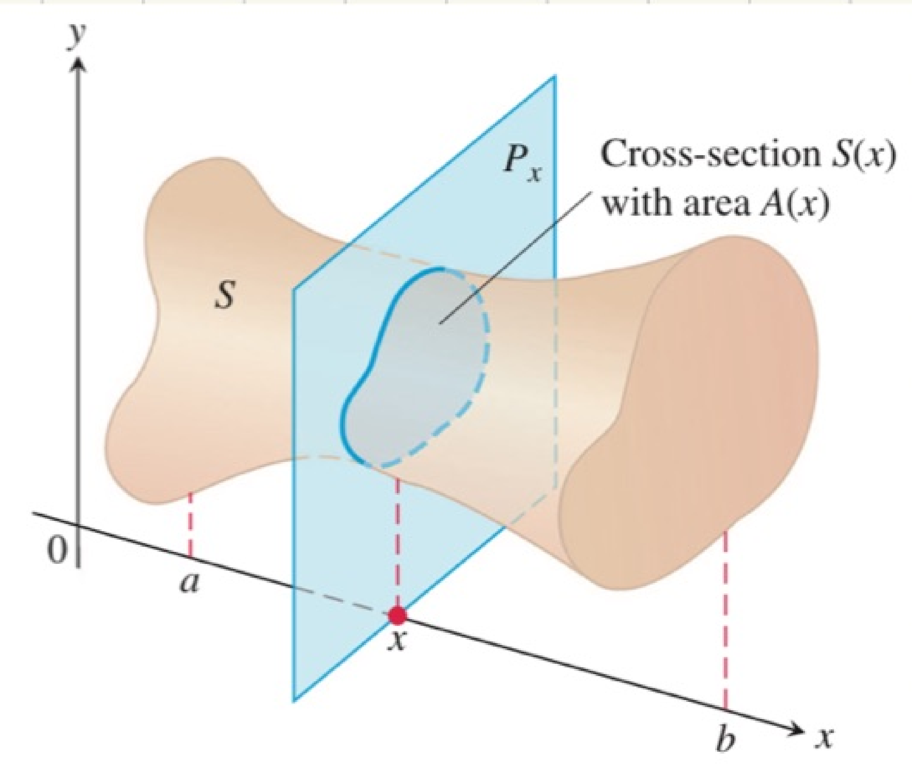


FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

Volumns of Solids of Revolution

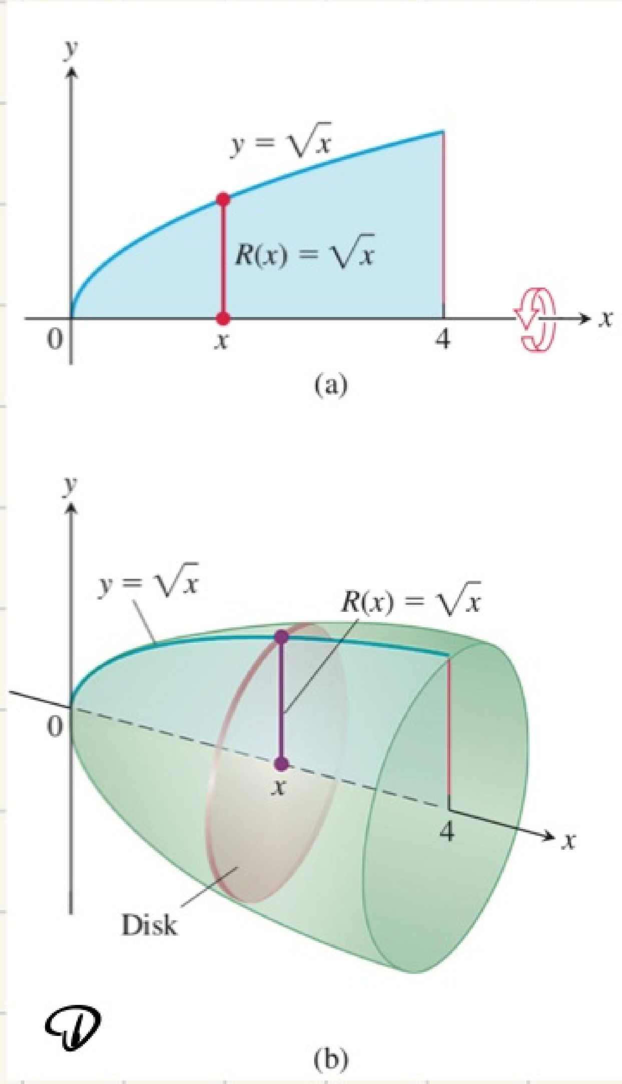
The disk method:

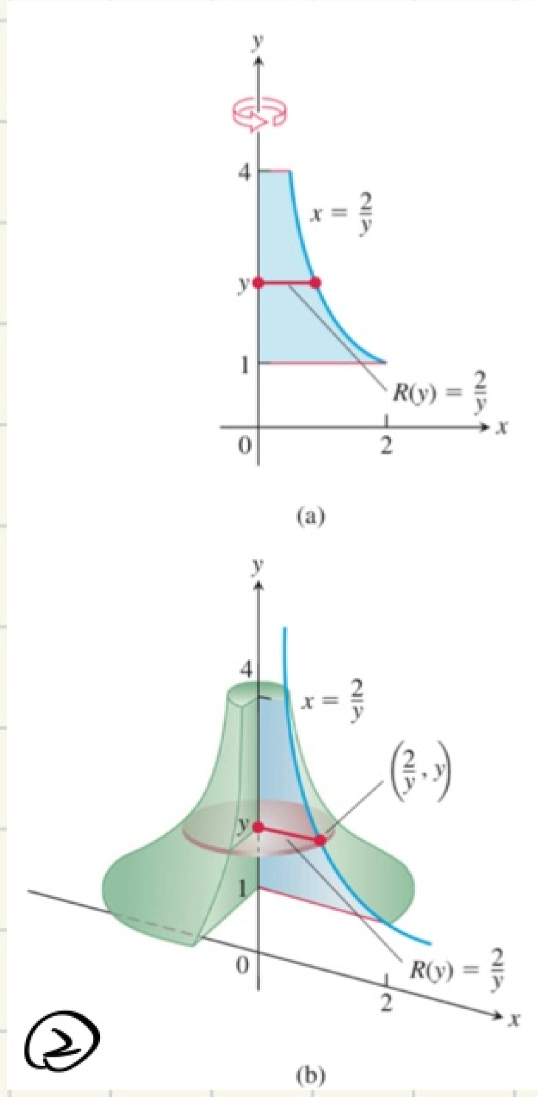
① Rotation about the x-axis:

$$V = \int_a^b A(x)\,dx = \int_a^b \pi [R(x)]^2\,dx$$

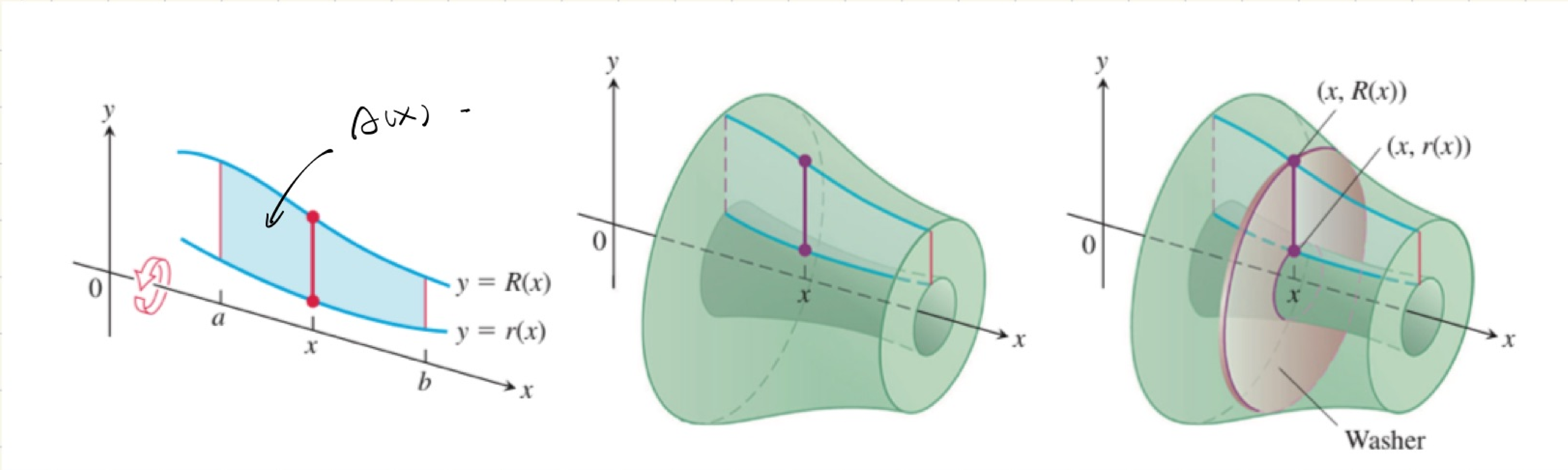
② Rotation about the y-axis:

$$V = \int_a^b A(y)\,dy = \int_a^b \pi [R(y)]^2\,dy$$





The Washer Method:



Actually similar to the disk method (Find the rotating $A(x)$)

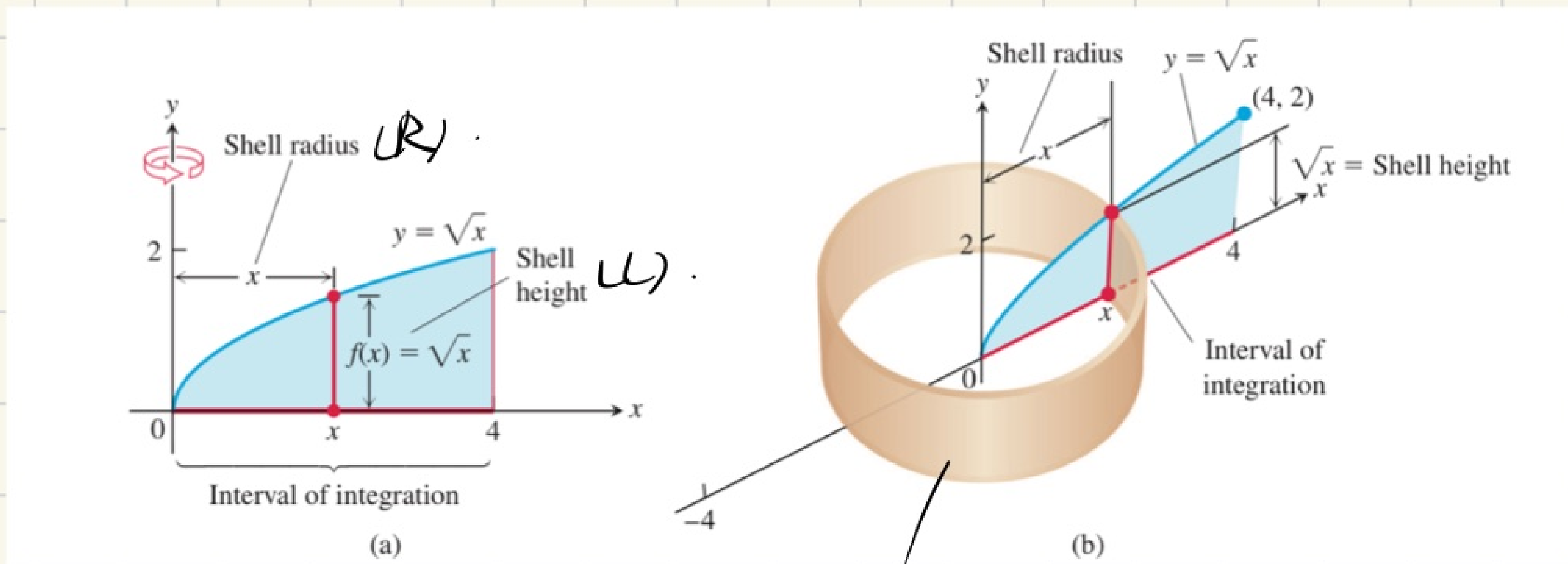
$$V = \int_a^b A(x)\,dx = \int_a^b \left[\pi R^2(x) - \pi r^2(x)\right] dx$$

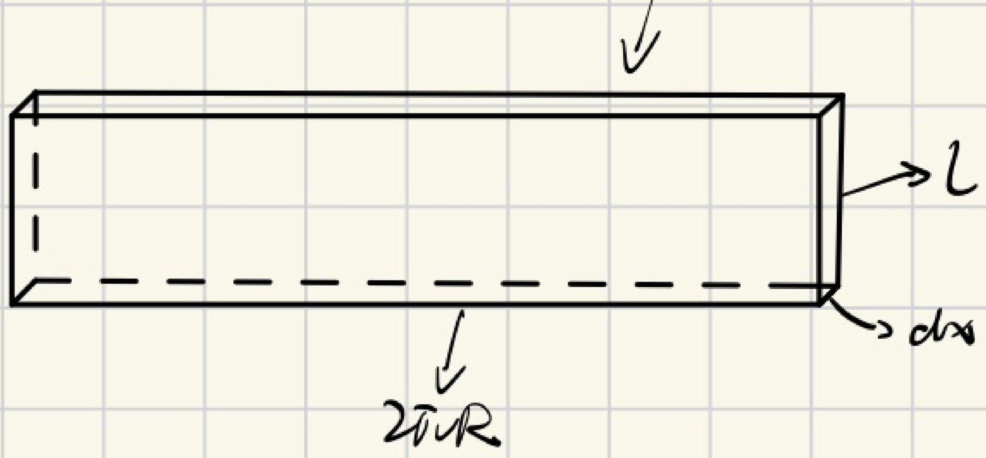
Volumns Using Cylindrical Shells

$$V = \int_a^b 2\pi R L\,dx$$

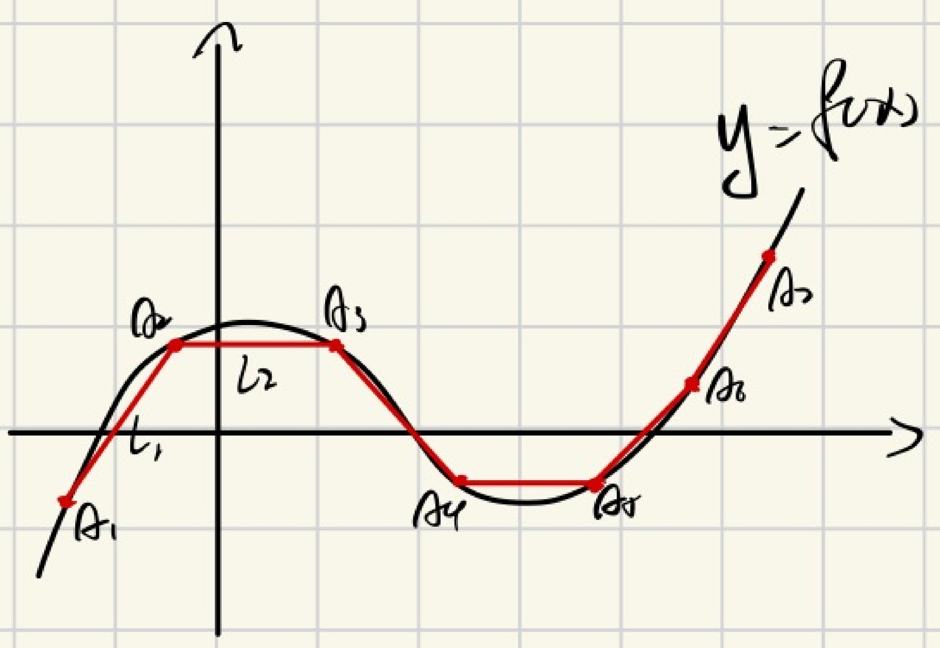
R: Shell Radius

L: Shell Height





Arc length



$L_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$, $\Delta y_k = f'(c_k)\Delta x_k$. $(x_{k-1} < c_k < x_k)$.

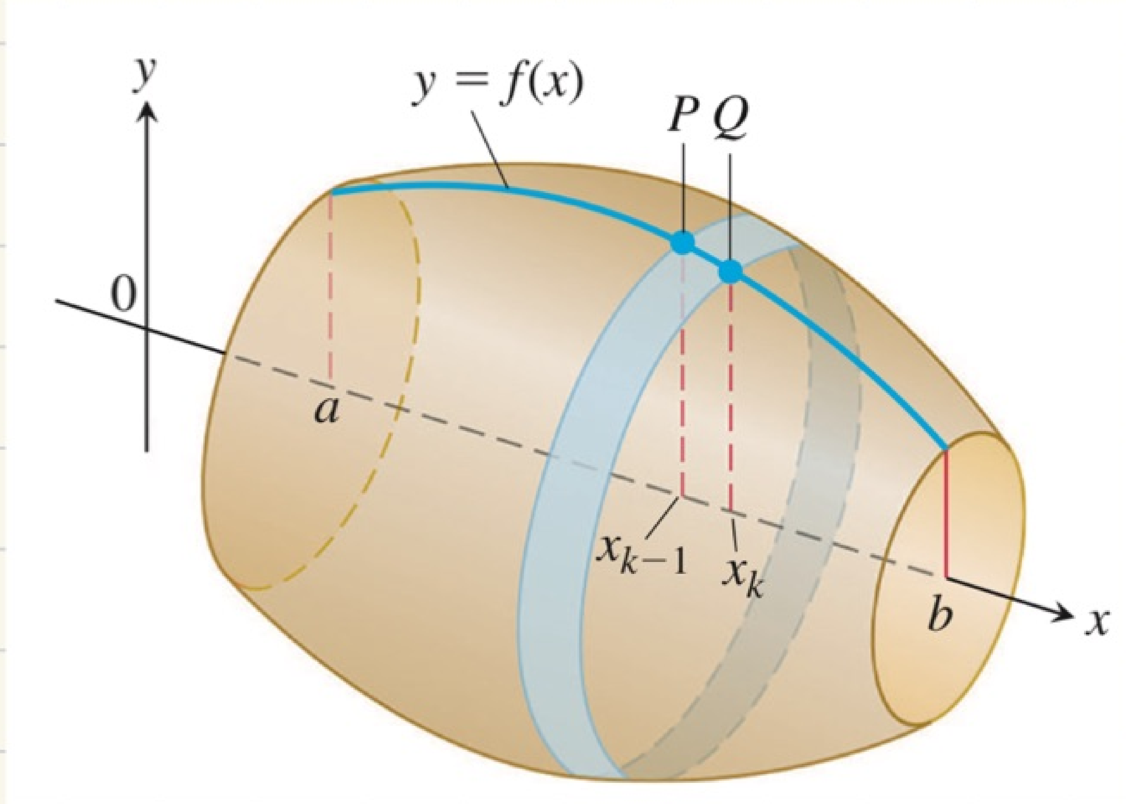
$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{\Delta x_k^2 + \Delta y_k^2} = \sum_{k=1}^{n} \sqrt{\Delta x_k^2 + f'(c_k)^2 \cdot \Delta x_k^2} = \sum_{k=1}^{n} \sqrt{1 + f'(c_k)^2} \cdot \Delta x_k$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + f'(c_k)^2} \cdot \Delta x_k = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$

$$\Rightarrow \star\ L = \int_a^b \sqrt{1 + f'(x)^2}\,dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

* $x = g(y)$ is similar & sometimes easier to compute!

- Areas of Surfaces of Revolution



Revolution about the x-axis:

$$S = \int_a^b 2\pi \cdot y \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx = \int_a^b 2\pi \cdot f(x) \cdot \sqrt{1+f'(x)^2}\, dx$$

* y-axis is similar

- Applications of Integer

1. Work done by a variable force along a line. ($W = F d$)

$$W = \int_a^b F(x)\, dx$$

2. Hooke's law for springs ($F = kx$).

3. Lifting objects and pumping liquids from containers.

4. Fluid pressure and forces

The Integral for Fluid Force Against a Vertical Flat Plate

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\, dy. \qquad (7)$$

III. Transcendental Functions

· One-to-one Functions

Def. A function $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

The horizontal line test:
$f(x)$ is one-to-one if & only if its graph intersects each horizontal line at most once

· Inverse Functions

*only oto can have inverse.

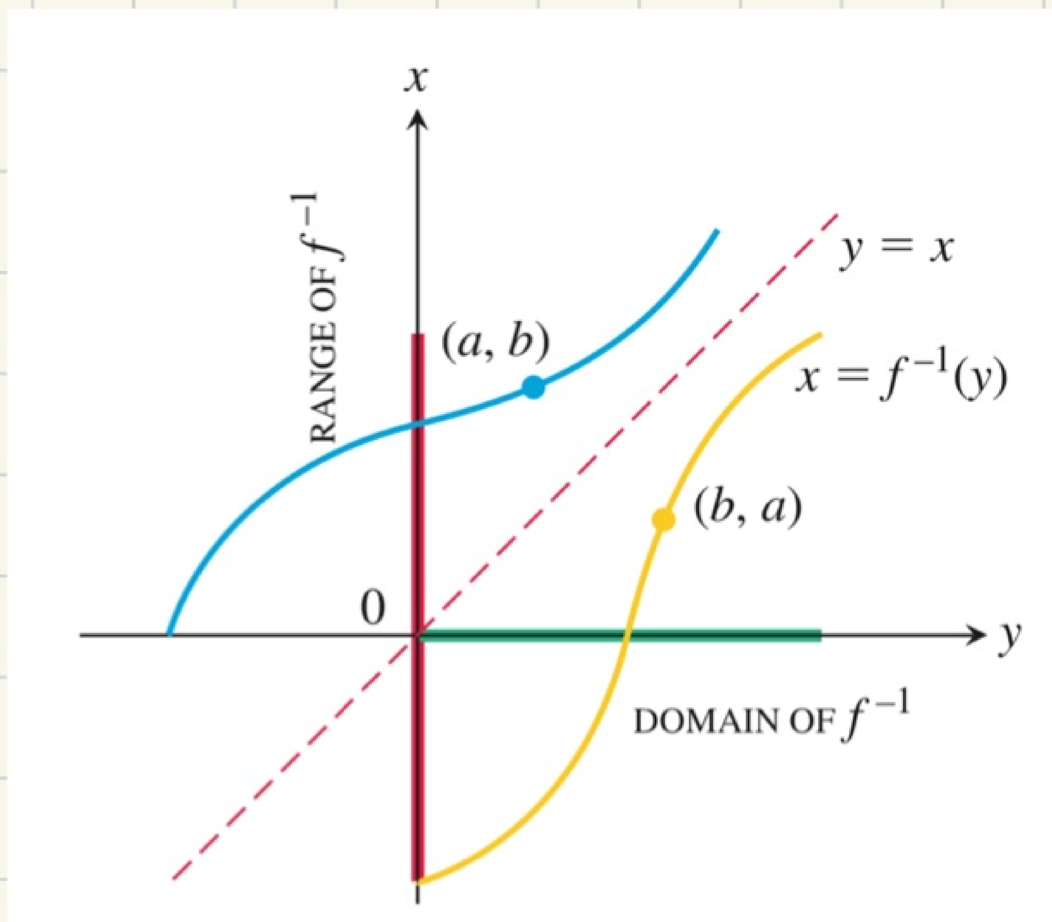
Def. Suppose that f is a one-to-one function on a domain D with range R.
The inverse function f^{-1} is defined by
$f^{-1}(b) = a$ if $f(a) = b$.
f^{-1}'s domain is R, range is D.

We have: $f^{-1}(f(x)) = x$ / $f(f^{-1}(x)) = x$

*Finding Inverses.

1. Solve the equation $y = f(x)$. for x.
we get $x = f^{-1}(y)$.

2. Interchange x & y. we get $y = f^{-1}(x)$.



e.g. Find the inverse of $y = \frac{1}{2}x + 1$.
$y = \frac{1}{2}x + 1 \Rightarrow x = 2y - 2$.
$\Rightarrow f^{-1}(x) = y = 2x - 2$.

· Derivative of Inverses

Theorem: The derivative rule for inverses.

$$\boxed{(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}}$$

or $$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Proof: $f(f^{-1}(x)) = x$

$\frac{d}{dx} f(f^{-1}(x)) = 1$.

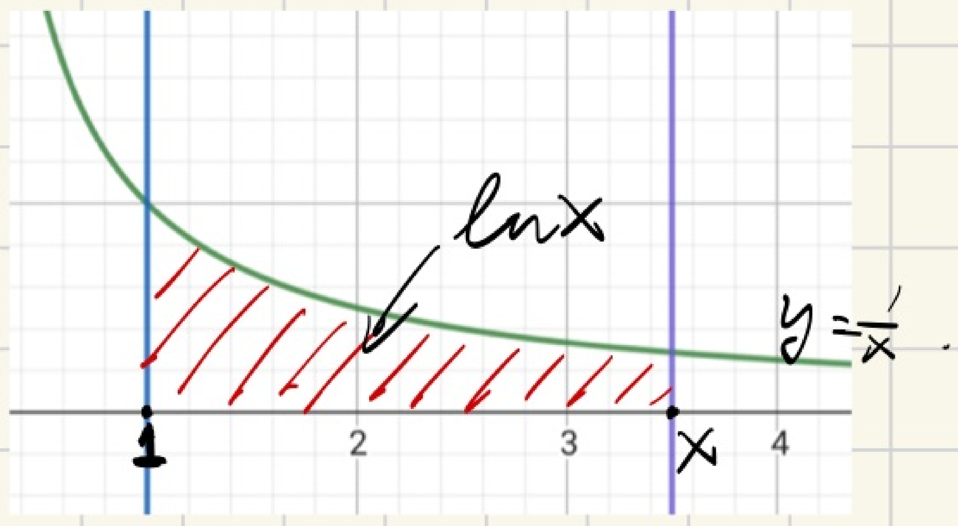
$\frac{d}{dx} f(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$ Chain Rule

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

Natural Logarithms

Def. The natural logarithm is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad (x>0).$$



Def. The number e is defined as:

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

Derivative: $\frac{d}{dx}\ln(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dx = \frac{1}{x}$.

Properties of Natural Logarithm (Algebraic Rules)

1. $\ln bx = \ln b + \ln x$ Product Rule
2. $\ln \frac{b}{x} = \ln b - \ln x$ Quotient Rule
3. $\ln x^r = r\ln x \ (r \in \mathbb{R})$ Power Rule

Indefinite Integral of $\frac{f'(x)}{f(x)}$

$$\int \frac{1}{x}\,dx = \ln|x| + C \quad (x \neq 0).$$

Proof: $\frac{d}{dx}\ln|x| = \frac{1}{|x|}\cdot\frac{|x|}{x} = \frac{1}{x}$

$$|x|' = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases} \rightarrow |x|' = \frac{|x|}{x}$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \quad (f(x) \neq 0).$$

Proof: $\frac{d}{dx}\ln|f(x)| = \frac{1}{|f(x)|}\cdot\frac{|f(x)|}{f(x)}\cdot f'(x) = \frac{f'(x)}{f(x)}$

Indefinite Integral of $\tan x$, $\cot x$, $\sec x$, $\csc x$

$$\int \tan x\,dx = \ln|\sec x| + C.$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

- **Logarithmic Differentiation**

To simplify the calculating differential of complex functions like:

$$F(x) = \prod_{k=1}^{n} (f_k(x))^{m_k} = (f_1(x))^{m_1} \cdot (f_2(x))^{m_2} \cdots (f_n(x))^{m_n} \quad (f_k(x) > 0)$$

Let $y = F(x)$. then

$$\ln y = m_1 \ln f_1(x) + m_2 \ln f_2(x) + \cdots + m_n \ln f_n(x).$$

$$\frac{d}{dx} \hookrightarrow \quad \frac{y'}{y} = m_1 \frac{f_1'(x)}{f_1(x)} + m_2 \frac{f_2'(x)}{f_2(x)} + \cdots + m_n \frac{f_n'(x)}{f_n(x)}.$$

$$y' = y \cdot \sum_{k=1}^{n} m_k \frac{f_k'(x)}{f_k(x)} \quad (f_k(x) > 0).$$

When $f_k(x) < 0$, replace $F(x)$ with $|F(x)|$.

s.t. $\frac{d}{dx} \ln|F(x)| = \frac{F'(x)}{F(x)}$. $\Rightarrow F'(x) = F(x) \cdot \frac{d}{dx} \ln|F(x)|$.

- **Exponential Functions**

Natural exponential function:

$$e^x = \underset{\ln^{-1} x}{\underline{\exp x}} \Rightarrow \begin{cases} e^{\ln x} = x & (x > 0) \\ \ln e^x = x & (x \in R). \end{cases}$$

Differential:

$$\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}.$$

Integral:

$$\int e^u du = e^u + C$$

Exponential function with base a:

$$a^x = e^{x \ln a} \quad (a > 0) \Rightarrow x^n = e^{n \ln x} \quad (\text{base } n \ \& \ x > 0).$$

- **Number e as a limit**

$$\boxed{e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}}$$

Proof: let $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. $f'(1) = 1$.

$$f'(1) = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x \to 0} \ln(1+x)^{\frac{1}{x}}$$

$$= \ln \left[\lim_{x \to 0} (1+x)^{\frac{1}{x}} \right]$$

Since $f'(1) = 1$, we have $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$.

- **The derivative of a^u**

  u is a differential function of x.

Differential:

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$$

Integral:

$$\int a^u du = \frac{a^u}{\ln a} + C$$

- **Logarithm with base a.**

Def. $\log_a x$ is the inverse function of a^x.

We have $a^{\log_a x} = x \quad (x>0)$

$\log_a(a^x) = x \quad (x \in R)$.

$$\log_a x = \frac{\ln x}{\ln a}.$$

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}\ln u = \frac{1}{\ln a \cdot u}\cdot\frac{du}{dx}$$

- **Indeterminate forms**

Def. Indeterminate forms are forms of "limit" which does not have a fixed value and cannot be determined by limit laws.

7 indeterminate forms:

e.g. 1) "$\frac{0}{0}$": as $x \to 0$, $\frac{\sin x}{x} \to 1$, $\frac{2x}{x} \to 2$.

2) "$\frac{\infty}{\infty}$": as $x \to \infty$, $\frac{2x}{x} \to 2$, $\frac{3x}{x} \to 3$.

3) $0 \cdot \infty$: as $x \to \infty$, $\frac{1}{x} \cdot x \to 1$, $\frac{1}{x} \cdot x^2 \to \infty$ ($\frac{1}{x} \cdot x^2 = x$)
$(1+\frac{1}{x}) \cdot x$, $1 \cdot \infty = \infty$

4) $\infty - \infty$: as $x \to \infty$, $x - x \to 0$, $2x - x \to \infty$, $x - x^2 \to -\infty$.

5) 0^0: as $x \to \infty$, $(e^{-x})^{\frac{1}{x}} \to e^{-1}$, $(e^{-x})^{\frac{2}{x}} \to e^{-2}$.

6) 1^∞: as $x \to 0^+$, $(1+x)^{\frac{1}{x}} \to e$, $(1+x)^{\frac{2}{x}} \to e^2$.

7) ∞^0: as $x \to 0^+$, $(e^{\frac{1}{x}})^{x} \to e$ ($e^{\frac{1}{x}} \to \infty$, $x \to 0$), $(e^{\frac{1}{x}})^{2x} \to e^2$.

Q: What about 0^∞?

A: It's not an indeterminate form.

If $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = \infty$.

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)}$$

$$= e^{\lim_{x\to a} g(x) \cdot \ln[\lim_{x\to a} f(x)]}$$

$$= e^{-\infty} = 0.$$

- **L'Hôpital's Rule**

Theorem: If $f(x)$, $g(x)$ is differential on an open interval containing a, and $\lim_{x\to a}\frac{f(x)}{g(x)}$ is like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ ($g(x)$ & $g'(x) \neq 0$), we have:

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

Proof: Using Cauchy's Mean Value Theorem.

Limits of products and quotients

Suppose that $\lim_{x\to a}\frac{f(x)}{g(x)} = L \in \mathbb{R}$, then:

$$\lim_{x\to a} f(x)\cdot h(x) = \lim_{x\to a}\frac{f(x)}{g(x)}\cdot g(x)\cdot h(x) = L\cdot \lim_{x\to a} g(x)\cdot h(x)$$

Remark: If $\lim_{x\to a}\frac{f(x)}{g(x)} = 1$, we may write $f(x)\sim g(x)$ as $x\to a$.

e.g. $\sin x \sim x$ as $x\to 0$, since $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

$e^x - 1 \sim x$ as $x\to 0$. since $\lim_{x\to 0}\frac{e^x-1}{x} = 1$.

e.g. Compute $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3} = ?$

$$LHS = \lim_{x\to 0}\frac{\sin x\cdot\frac{1-\cos x}{\cos x}}{x^3} = \lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3}\cdot\frac{1}{\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1-\cos x}{\frac{1}{2}x^2}\cdot\frac{1}{2}\cdot\frac{1}{\cos x}.$$

$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2}\cdot\lim_{x\to 0}\frac{1}{2}\cdot\lim_{x\to 0}\frac{1}{\cos x} = 1\cdot 1\cdot\frac{1}{2}\cdot 1 = \frac{1}{2}.$$

Relative Rates of Growth

Def. Let $f(x)$ & $g(x)$ be positive for x sufficiently large.

1. f grows faster than g as $x\to\infty$ if
$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \infty$$
f grows slower than g as $x\to\infty$ if
$$\lim_{x\to\infty}\frac{g(x)}{f(x)} = 0$$

2. f & g grow at the same rate as $x\to\infty$ if
$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L \qquad (L \text{ is finite \& positive}).$$

Order and Oh-Notation

Def. A function f is of smaller order than g as $x\to\infty$ if:
$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$ (f grows slower than g).

Notation: $f = o(g)$. ("f is little-oh of g")

e.g. $x^2 = o(x^3+1)$ as $x\to\infty$, because $\lim_{x\to\infty}\frac{x^2}{x^3+1} = 0$.

Def. Let $f(x)$ & $g(x)$ be positive for x sufficiently large. Then f is of at most the order of g as $x\to\infty$ if:
$\frac{f(x)}{g(x)} \le M$ for f sufficiently large. ($M \in \mathbb{Z}^+$)

Notation: $f = O(g)$ ("f is big-oh of g")

e.g. $x+\sin x = O(x)$, because $\frac{x+\sin x}{x} \le 2$ for x sufficiently large

Inverse Trigonometric Functions

Def.

			Range
$\sin y = x$	$\Rightarrow$	$y = \arcsin x = \sin^{-1}x$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos y = x$	$\Rightarrow$	$y = \arccos x = \cos^{-1}x$	$[0, \pi]$
$\tan y = x$	$\Rightarrow$	$y = \arctan x = \tan^{-1}x$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\cot y = x$	$\Rightarrow$	$y = \operatorname{arccot} x = \cot^{-1}x$	$(0, \pi)$
$\sec y = x$	$\Rightarrow$	$y = \operatorname{arcsec} x = \sec^{-1}x$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$\csc y = x$	$\Rightarrow$	$y = \operatorname{arccsc} x = \csc^{-1}x$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Derivatives

1. $\frac{d}{du}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$
2. $\frac{d}{du}(\cos^{-1}u) = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$
3. $\frac{d}{du}(\tan^{-1}u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$
4. $\frac{d}{dx}(\cot^{-1}u) = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$
5. $\frac{d}{dx}(\sec^{-1}u) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$
6. $\frac{d}{dx}(\csc^{-1}u) = -\frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

Proof ①

Let $f(x) = \sin x$ & $f^{-1}(x) = \sin^{-1}(x)$.

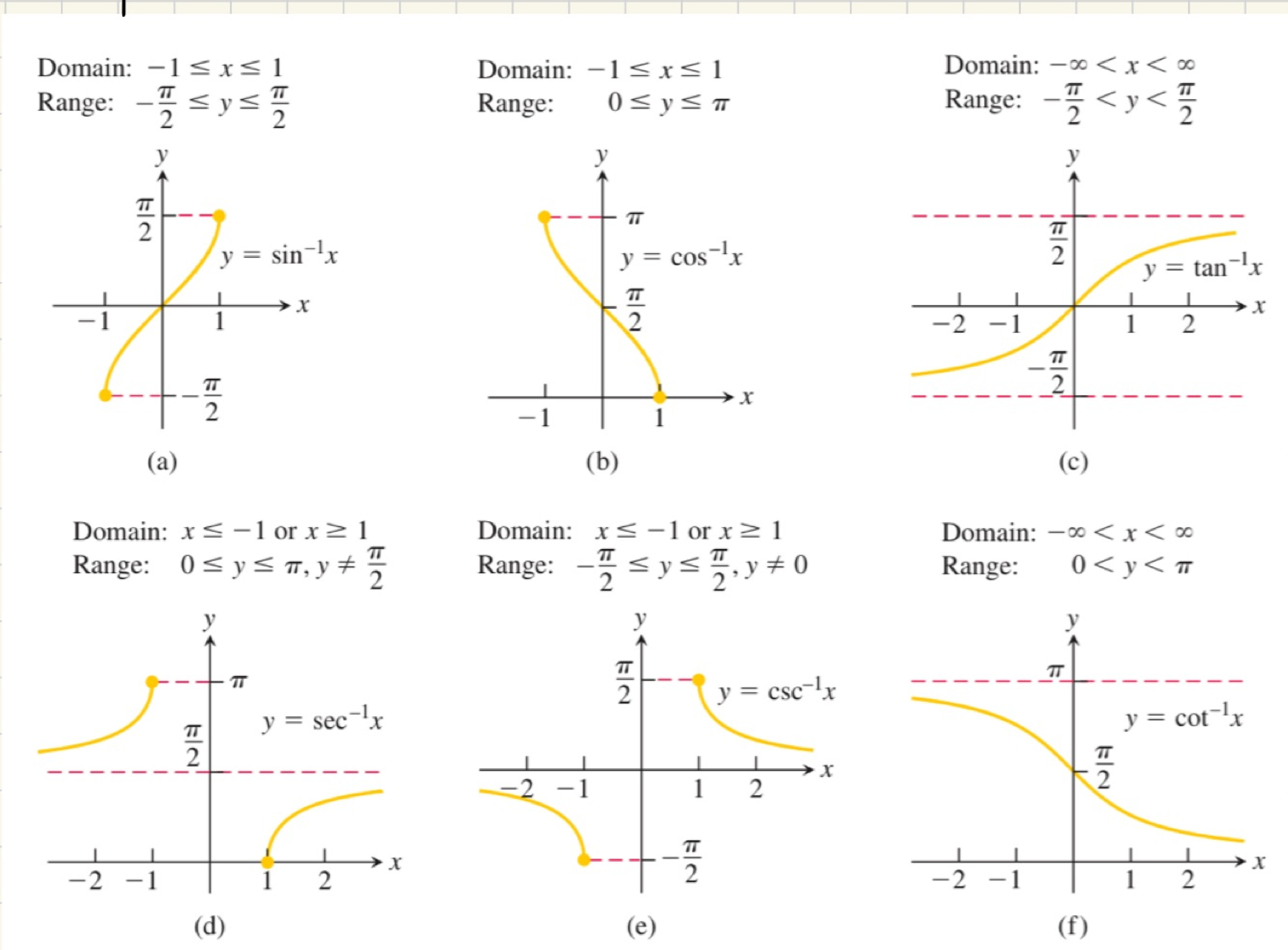
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{1-\sin^2(\sin^{-1}x)}} = \frac{1}{\sqrt{1-x^2}}$$

Others are similar.

Related Integrals

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C, \quad u^2 < a^2$
2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C, \quad |u| > a > 0$

Graph



IV Techiques of Integration

- **Integration by Parts**

Formula:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Another form: $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

e.g. $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$.

$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C$.

Reduction formula:

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1}x\sin x + (n-1)\int \sin^2 x\cos^{n-2}x\,dx \\ &= \cos^{n-1}x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,dx \\ &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx \\ &= \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx.\end{aligned}$$

Definite integrals.

$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx}$$

e.g. $\int_0^4 xe^{-x}dx = -xe^{-x}\Big]_0^4 - \int_0^4(-e^{-x})dx = 1-5e^{-4}$

Tabular integration

$\int x^2e^x dx = x^2e^x - 2xe^x + 2e^x + C$

$f(x)$ & $f'(x)$		$g(x)$ & $\int g(x)$
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

- **Trigonometric Integrals**

▷ Form $\int \sin^m x\cos^n x\,dx \Rightarrow$ transfer into all sin or all cos.

1) m is odd, let $m=2k+1$.
   $\sin^m x = (\sin^2 x)^k\sin x = (1-\cos^2 x)^k\sin x$.
   Combine $\sin x$ with dx, $\sin x\,dx = -d(\cos x)$.

2) m is even & n is odd, let $n=2k+1$
   $\cos^n x = (\cos^2 x)^k\cos x = (1-\sin^2 x)^k\cos x$
   Combine $\cos x$ with dx, $\cos dx = d(\sin x)$.

3) Both m & n are even $\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$.

e.g. $\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x \sin x\,dx$

$= \int (1-\cos^2 x)(\cos^2 x)(-d(\cos x))$

$= \int (1-u^2)(u^2)(-du)$ Let $u = \cos x$

$= \int (u^4 - u^2)\,du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{5} + C$

∘ Form $\int \tan^m x \sec^n x\,dx$

We use $\tan^2 x = \sec^2 x - 1$, $\sec^2 x = \tan^2 x + 1$

e.g. $\int \tan^4 x\,dx = \int \tan^2 x \cdot \tan^2 x\,dx$

$= \int \tan^2 (\sec^2 x - 1)\,dx$

$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$

$= \int \tan^2 x \sec^2 x\,dx^{*} - \int (\sec^2 x - 1)\,dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

* Let $u = \tan x$, $du = \sec^2 dx$

$\int \tan^2 x \sec^2 dx$

$= \int u^2\,du$

$= \frac{1}{3}u^3 + C$

∘ Products of sin & cos

Use identities:

$$\left.\begin{aligned} \sin mx \sin nx &= \tfrac{1}{2}[\cos(m-n)x - \cos(m+n)x] \\ \sin mx \cos nx &= \tfrac{1}{2}[\sin(m-n)x + \sin(m+n)x] \\ \cos mx \cos nx &= \tfrac{1}{2}[\cos(m-n)x + \cos(m+n)x] \end{aligned}\right\}$$

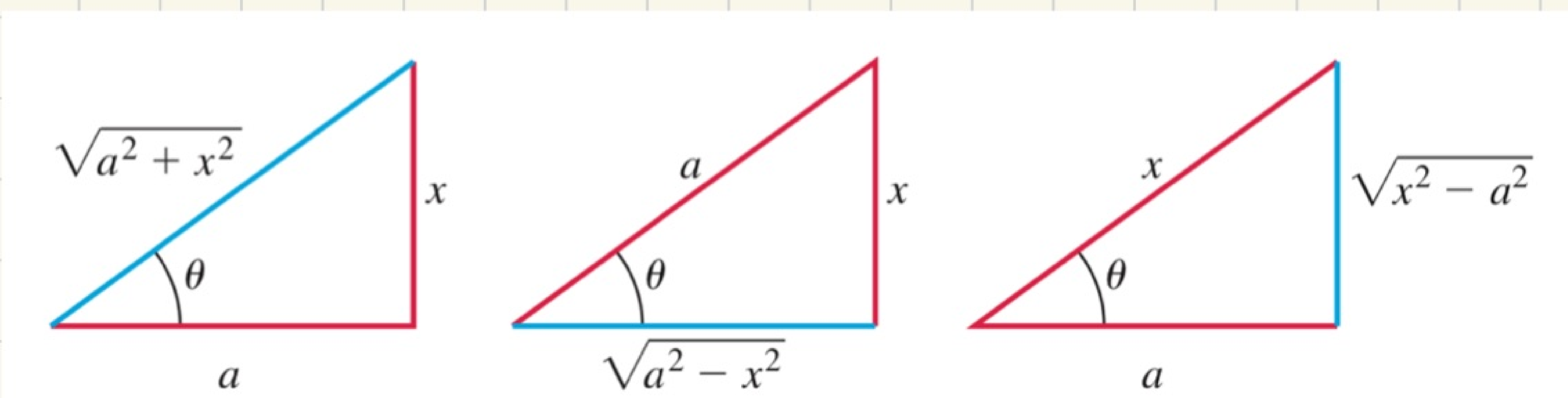
* product to sum

e.g. $\int \sin 3x \cos 5x\,dx = \frac{1}{2}\int [\sin(-2x) + \sin 8x]\,dx$

$= \frac{1}{2}\int (\sin 8x - \sin 2x)\,dx$

$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C$

· Trigonometric Substitutions



$x = a\tan\theta$ $x = a\sin\theta$ $x = a\sec\theta$

$\sqrt{a^2+x^2} = a|\sec\theta|$ $\sqrt{a^2-x^2} = a|\cos\theta|$ $\sqrt{x^2-a^2} = a|\tan\theta|$

☆ 1° $x = a\tan\theta$, $x^2 + a^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$ $(\theta = \tan^{-1}(\frac{x}{a}) \in (-\frac{\pi}{2}, \frac{\pi}{2}))$

2° $x = a\sin\theta$, $a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$ $(\theta = \sin^{-1}(\frac{x}{a}) \in (-\frac{\pi}{2}, \frac{\pi}{2}))$

3° $x = a\sec\theta$, $x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$ $(\theta = \sec^{-1}(\frac{x}{a}) \begin{cases} \in [0, \frac{\pi}{2}), \text{若 } \frac{x}{a} \geq 1. \\ \in (\frac{\pi}{2}, \pi], \text{若 } \frac{x}{a} \leq -1. \end{cases}$

e.g. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}}^{*} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|}$

$\displaystyle= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$\displaystyle= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$

* Set $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$4+x^2 = 4+4\tan^2\theta = 4\sec^2\theta$

- Integrations of Rational Functions by Partial Fractions

e.g. $\displaystyle\int \frac{dx}{x^2-x-12} = \int \frac{dx}{(x-4)(x+3)} = \frac{1}{7}\int \left(\frac{1}{x-4} - \frac{1}{x+3}\right) dx = \frac{1}{7}\left(\int \frac{dx}{x-4} - \int \frac{dx}{x+3}\right) = \frac{1}{7}\ln|x-4| - \frac{1}{7}\ln|x+3| + C$

Rational Function　　　　Partial Function

$\displaystyle= \frac{1}{7}\ln\left|\frac{x-4}{x+3}\right| + C$

Theorem: Consider rational function $\frac{p(x)}{q(x)}$ ($p(x)$ & $q(x)$ are polynomials).

Suppose · degree of $p(x)$ < degree of $q(x)$.

· no common factors of $p(x)$ & $q(x)$.

Then · $q(x)$ can be factorized into the following form:

$q(x) = c\,(x-r_1)^{m_1}\cdot(x-r_2)^{m_2}\cdots(x-r_k)^{m_k}\cdot(x^2+b_1x+c_1)^{n_1}\cdot(x^2+b_2x+c_2)^{n_2}\cdots(x^2+b_lx+c_l)^{n_l}.$

where $(x^2+b_1x+c_1)$, $(x^2+b_2x+c_2)\cdots$ are irreducible ($\Delta = b^2-4ac < 0$).

· $\frac{p(x)}{q(x)}$ can be decomposed as sum of "partial fractions".

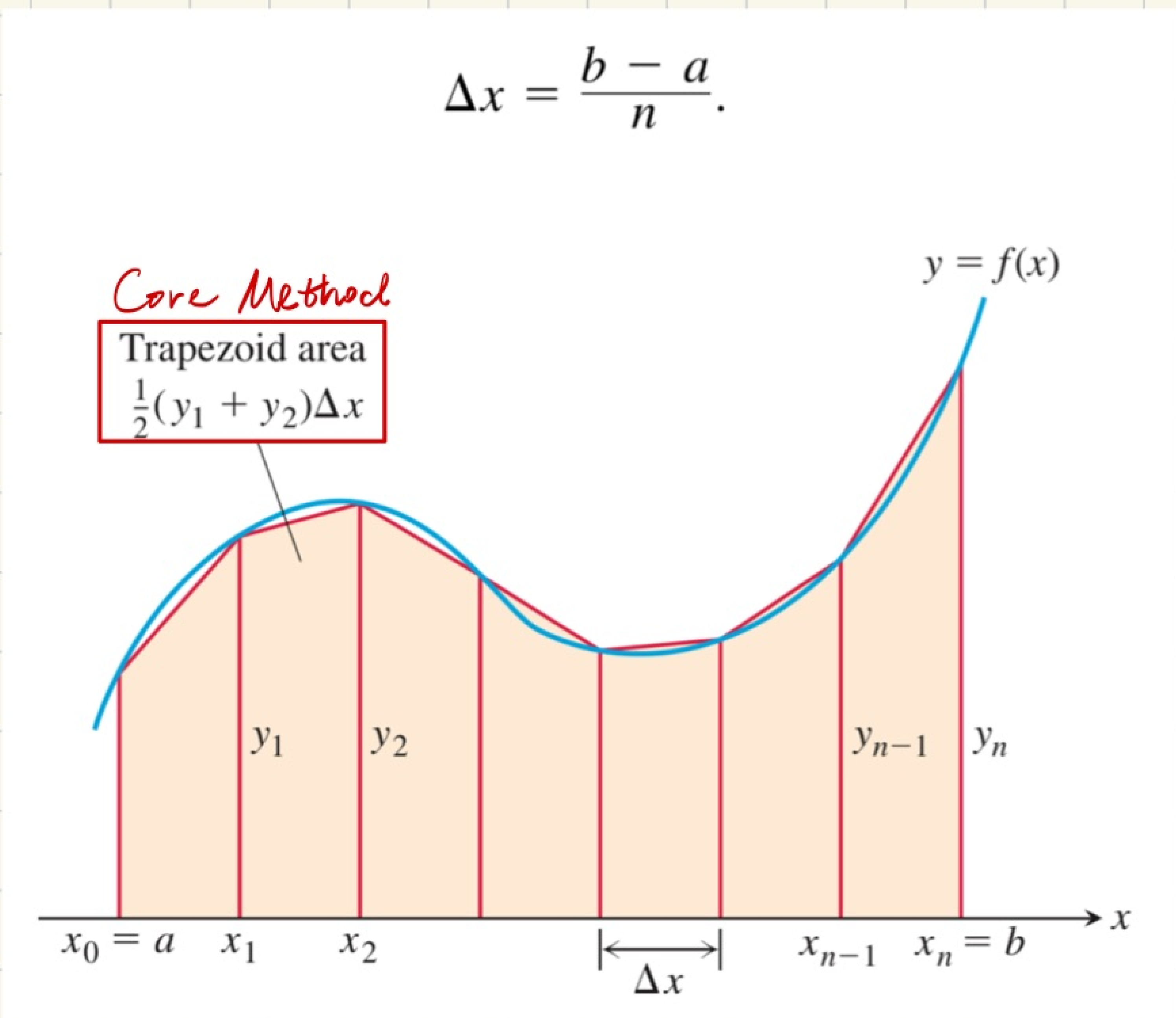
$$\frac{p(x)}{q(x)} = \frac{A_{11}}{x-r_1} + \frac{A_{12}}{(x-r_1)^2} + \cdots + \frac{A_{1m_1}}{(x-r_1)^{m_1}} + \cdots + \frac{A_{k1}}{x-r_k} + \frac{A_{k2}}{(x-r_k)^2} + \cdots + \frac{A_{km_k}}{(x-r_k)^{m_k}} +$$
$$\frac{B_{11}x+C_{11}}{x^2+b_1x+c_1} + \cdots + \frac{B_{1n_1}+C_{1n_1}}{(x^2+b_1x+c_1)^{n_1}} + \cdots + \frac{B_{l1}x+C_{l1}}{x^2+b_lx+c_l} + \cdots + \frac{B_{ln_l}x+C_{ln_l}}{(x^2+b_lx+c_l)^{n_l}}$$

$\forall x \neq r_1, r_2, \ldots r_k$. A, B, C const.

e.g. ?

- Numerical Integration

① Trapezoidal Approximations



To approximate $\int_a^b f(x)\,dx$, use:

$\displaystyle \circ\, T_n = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}\,\Delta x$

$\displaystyle= \frac{f(x_0)+f(x_1)}{2}\Delta x + \frac{f(x_1)+f(x_2)}{2}\Delta x + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\Delta x$

$\displaystyle= \frac{b-a}{2n}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$

$\displaystyle= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$

② Simpson's Rule: Approximations using parabolas

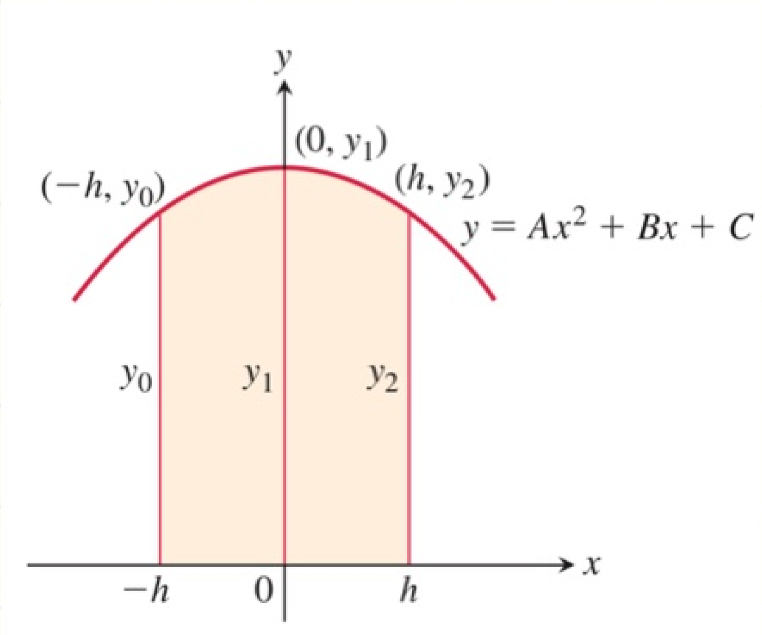
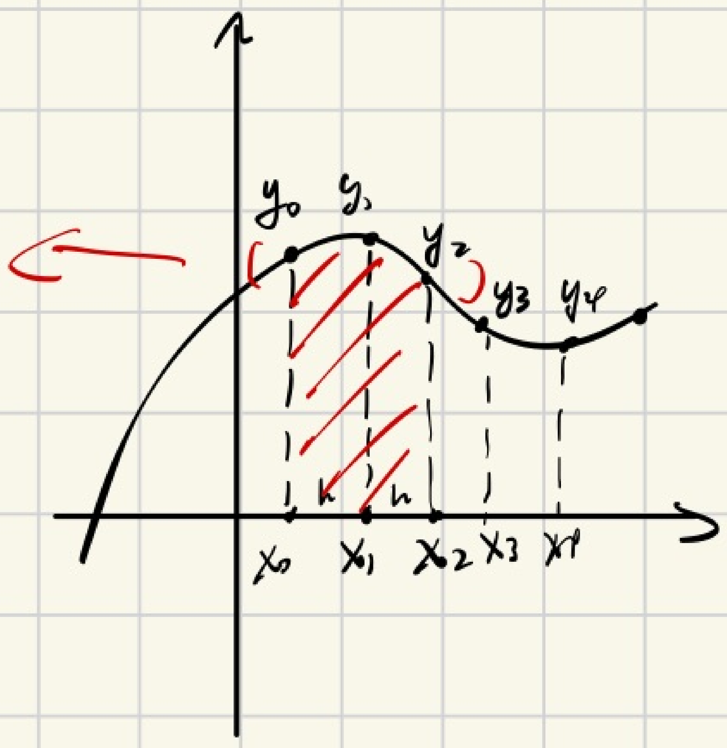


FIGURE 8.10 By integrating from $-h$ to h, we find the shaded area to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$



Let $h = \Delta x = \frac{b-a}{n}$.

We choose continous 3 pts,

& 3 pts must form a parabola if they are not on a straight line.

Area of the parabola is

$$A_p = \int_{-h}^{h} (Ax^2+Bx+C)\,dx = \frac{h}{3}(2Ah^2+6C)$$

We put $(-h, y_0)$, $(0, y_1)$, (h, y_2) into it.

& get $A_p = \frac{h}{3}(y_0 + 4y_1 + y_2)$

Thus. $\int_a^b f(x)dx \approx \Sigma A_p = \frac{h}{3}(y_0+4y_1+y_2) + \frac{h}{3}(y_2+4y_3+y_4) + \cdots + \frac{h}{3}(y_{n-2}+4y_{n-1}+y_n)$

$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$

Warning: n has to be even! $(2|n)$.

③ Error?

Riemman Sum Approximate $E_L \le O(\frac{1}{n})$. $E_R \le O(\frac{1}{n})$. $E_m \le O(\frac{1}{n^2})$.

$$E_T \le \frac{M(b-a)^3}{12n^2} = O\left(\frac{1}{n^2}\right).$$

$$E_S \le \frac{M(b-a)^5}{180n^4} = O\left(\frac{1}{n^4}\right) \Rightarrow \text{best way of approximate definite integrals.}$$

· **Improper Integrals**

Def. $\int_a^b f(x)dx$ is said to be improper if either $[a,b]$ is infinite, or $f(x)$ becomes unbounded.

∘ Type I.

① $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)dx$ ($f(x)$ is contin. on $[a,+\infty)$)

② $\int_{-\infty}^a f(x)dx = \lim_{b\to-\infty}\int_b^a f(x)dx$ ($f(x)$ is contin. on $(-\infty,a]$)

③ $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$ ($f(x)$ is contin. on $\mathbb{R}$).

e.g. $\int_a^\infty \frac{1}{x^4}dx = \lim_{b\to\infty}\int_a^b x^{-4}dx = \lim_{b\to\infty}\left[-\frac{1}{3}x^{-3}\right]_a^b = \frac{1}{3}a^{-3}$ (s.t. $\int_a^\infty \frac{1}{x^4}dx$ converges)

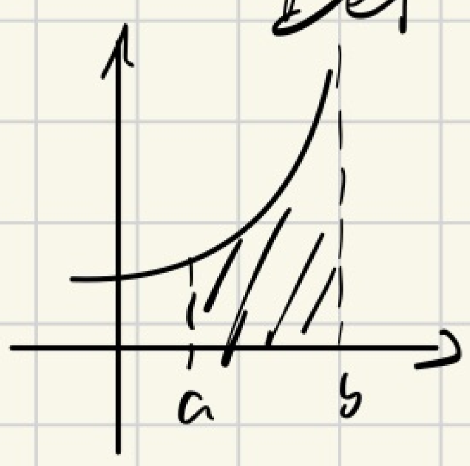
e.g. $\int_{-\infty}^\infty \sin x\,dx = \int_{-\infty}^0 \sin x\,dx + \int_0^\infty \sin x\,dx$

Since $\int_0^\infty \sin x dx = \lim_{b\to\infty}\int_0^b \sin x dx = \lim_{b\to\infty}[-\cos x]_0^b = \lim_{b\to\infty}(-\cos b+1)$ DNE!!

$\int_{-\infty}^\infty \sin x dx$ diverges. (no meaning to talk about the value).

○ Type II

Def. Consider $\int_a^b f(x)dx$, $[a,b]$ finite, $f(x)$ becomes unbounded near b.



If $\lim_{c\to b}\int_a^c f(x)dx$ exists, then we say $\int_a^b f(x)dx$ converges & $= \lim_{b\to c}\int_a^c f(x)dx$.

① $\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)dx$ ($f(x)$ is contin. on $(a,b]$ & discontin. at a).

② $\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)dx$ ($f(x)$ is contin. on $[a,b)$ & discontin. at b).

③ $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ ($f(x)$ is discontin. at c & contin. on $[a,c)\cup(c,b]$)

e.g. $\int_0^1 \frac{1}{1-x}dx = \lim_{b\to 1^-}\int_0^b \frac{1}{1-x}dx = \lim_{b\to 1^-}\left[-\ln|1-x|\right]_0^b = \lim_{b\to 1^-}\left[-\ln|1-b| + \ln 1\right] = \infty$

Thus, the integral diverges.

· Convergence or Divergence ?

Theorem

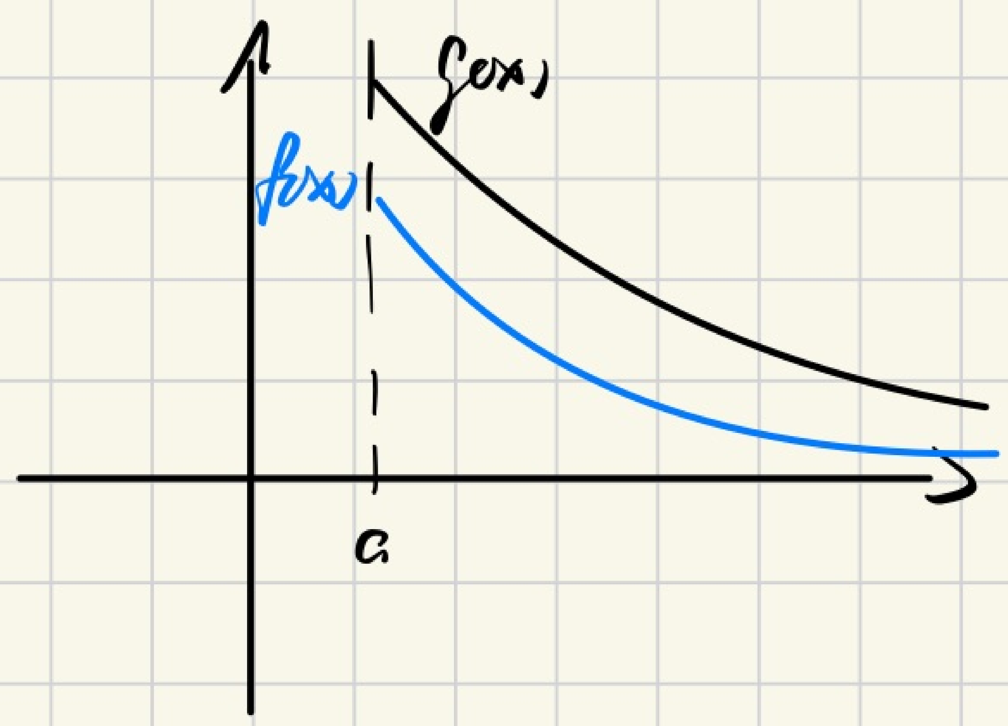
1. If improper $\int_a^b |f(x)|\,dx$ converges (diverges), then $\int_a^b f(x)dx$ converges (diverges).

2. Direct Comparison Test

$0 \le f(x) \le g(x)$, $\forall x \in (a,b)$.

① If $g(x)$ converges, $f(x)$ converges.

② If $f(x)$ diverges, $g(x)$ diverges.



3. Limit Comparison Test

$0 \le f(x) \le g(x)$, $\forall x \in [a,\infty)$.

Suppose $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$ exists & $= L$ (maybe $= \infty$)

① If $0 < L < \infty$, then $\int_a^\infty f(x)dx$ & $\int_a^\infty g(x)dx$ converge and diverge at the same time

② If $L = 0$, If $\int_a^\infty g(x)dx$ converges, $\int_a^\infty f(x)dx$ converges.
If $\int_a^\infty f(x)\,dx$ diverges, $\int_a^\infty g(x)\,dx$ diverges

③ If $L = \infty$ If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.
If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.

V. First-Order Differential Equations

General First-Order Differential Equations

Def. A first-order diff eq. is an eq. of the form:

$$\boxed{\frac{dy}{dx} = f(x, y)} \quad ①$$

Def. A solution of ① is a function $y(x)$ such that:

$$\frac{dy(x)}{dx} = f(x, y(x)).$$

The general solution of ① is the collection of all possible solutions of ①.

e.g. $\frac{dp}{dt} = rp$ ($r > 0$, const). ② Malthusiem Model

Logestic eq: $\frac{dp}{dt} = rp(1 - \frac{P}{K})$. ☆

This is where your 11.5w in CUHKSZ values

$$\frac{dp}{dt} - rp = 0$$

$$e^{-rt} \cdot \frac{dp}{dt} - \underbrace{e^{-rt} \cdot rp}_{\frac{d}{dt}(e^{-rt})} = 0$$

$$e^{-rt} \cdot \frac{dp}{dt} + \frac{dp}{dt} \cdot e^{-rt} = 0$$

$$\frac{d}{dt}(e^{-rt} \cdot p) = 0$$

$\Rightarrow e^{-rt} \cdot p = C$ (const).

$\Rightarrow p = C \cdot e^{rt}$ (general sol. of ②).

Initial Value Problem

IVP: $\begin{cases} \frac{dy}{dx} = f(x, y). \\ y(x_0) = y_0 \end{cases}$ → initial value

x_0 → initial time.

$y(x_0) = y_0$ → initial condition.

particular sol. is a sol. of IVP.

eg. $\begin{cases} \frac{dp}{dt} = rp \\ P(0) = P_0 \end{cases}$

general sol. $p(t) = C \cdot e^{rt}$

$p(0) = C = p_0$.

$\Rightarrow p(t) = p_0 e^{rt}$

Two Solution Methods.

Goal: Find sols of diff. eg. by barehand.

1. Seperable eq.

▷ $\frac{dy}{dx} = \underline{f(x) \cdot g(y)}$ product!

$\begin{cases} e^{x+y} = e^x \cdot e^y \ ✓ \\ e^x + e^y \ ✗ \end{cases}$

$$\frac{dy}{g(y)} = f(x) dx.$$

$$\int \frac{dy}{g(y)} = \int f(x) dx \Rightarrow \text{solve } y(x)!$$

eg. $\frac{dy}{dx} = e^x \cdot e^y$.

$\frac{dy}{e^y} = e^x dx$.

$\int \frac{dy}{e^y} = \int e^x dx$

$-e^{-y} = e^x + C$

$\ln e^{-y} = \ln(-e^x - C)$.

$\Rightarrow y = -\ln(-e^x - C)$ general sol e.g. IVP $\begin{cases} \frac{dy}{dx} = xy^2 \\ y(0) = 0 \end{cases}$

$$\int \frac{dy}{y^2} = \int x\,dx$$ this means $y \neq 0$!

$$-\frac{1}{y} = \frac{1}{2}x^2 + C$$

$\Rightarrow y = -\frac{2}{x^2+C}$. But! $y(0) \overset{?}{=} 0$ ×

Thus. general sol. $\begin{cases} y = -\frac{2}{x^2+C} \\ y \equiv 0 \end{cases}$.

2. First-Order Linear Eq.

$$\frac{dy}{dx} + a(x)y = b(x).$$

$$I(x)\cdot\frac{dy}{dx} + I(x)\cdot a(x)\cdot y = I(x)\cdot b(x).$$

Want $\frac{dI}{dx} = a(x)\cdot I(x)$

$\frac{dI}{I} = a(x)\,dx$. Integration

$\ln|I| = \int a(x)dx$

$|I| = e^{\int a(x)dx}$

$$I(x)\cdot\frac{dy}{dx} + \frac{dI}{dx}\cdot y = I(x)\cdot b(x).$$

$$\frac{d}{dx}(I\cdot y) = I(x)\cdot b(x).$$

$$y = \frac{1}{I(x)}\int I(x)b(x)\,dx.$$

Conclusion: To solve $\frac{dy}{dx} + a(x)\cdot y = b(x)$, multiply $I(x) = e^{\int a(x)dx}$ by both sides, then integrate both sides.

e.g. $x\frac{dy}{dx} = x^2 + 3y$, $x > 0$

$$\frac{dy}{dx} - \frac{3y}{x} = x$$

We have $e^{\int -\frac{3}{x}dx} = e^{-3\ln|x|} = \frac{1}{x^3}$.

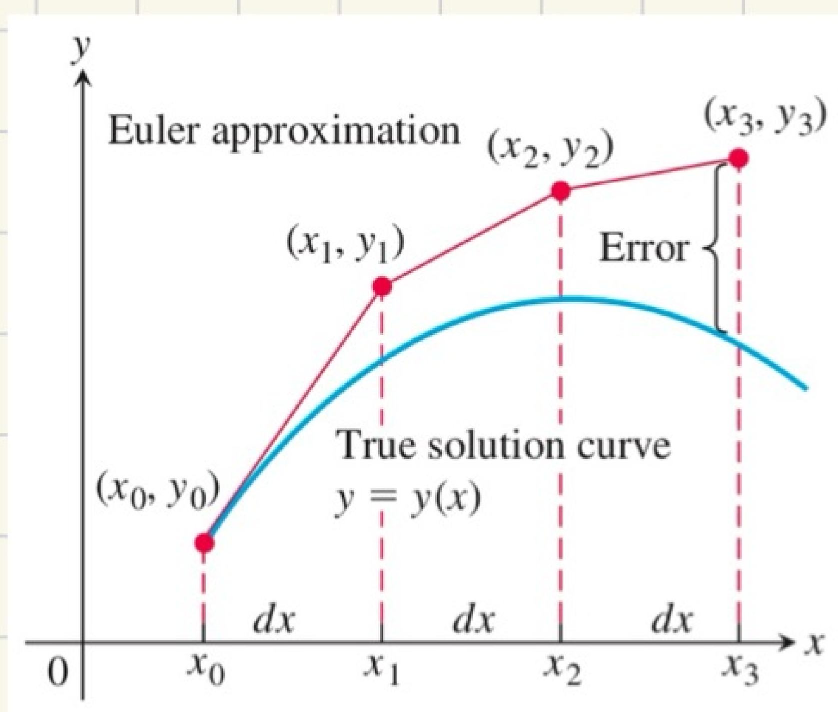
Multiply $\frac{1}{x^3}$ by both sides, and we get:

$$\frac{1}{x^3}\cdot\frac{dy}{dx} - \frac{1}{x^3}\cdot\frac{3y}{x} = \frac{1}{x^3}\cdot x$$

$$\frac{d}{dx}\left(\frac{1}{x^3}\cdot y\right) = \frac{1}{x^2}.$$

$$\frac{1}{x^3}\cdot y = \int \frac{1}{x^2}dx$$

$$y = \left(-\frac{1}{x} + C\right)\cdot x^3 = -x^2 + Cx^3,\quad x > 0.$$



- Euler's Method

Give $\frac{dy}{dx} = f(x,y)$ & $y(x_0) = y_0$, we can approximate $y = y(x)$ by linearization $L(x) = y(x_0) + y'(x_0)(x - x_0)$. or $L(x) = y(x_0) + f(x,y)\cdot(x - x_0)$.

Then $y_1 = L(x_1) = y_0 + f(x_0,y_0)\cdot(x_1 - x_0) = y_0 + f(x_0,y_0)\cdot\Delta x$

$y_2 = L(x_2) = y_1 + f(x_1,y_1)\cdot(x_2 - x_1) = y_1 + f(x_1,y_1)\cdot\Delta x$

......

when Δx gets small, it will get a good approximation of $y = y(x)$.

- Autonomous Equations

Def. Differential Eqs like $\frac{dy}{dx} = g(y)$ is an autonomous eq.
The values of y for which $\frac{dy}{dx} = 0$ are called equilibrium values or rest points.

Phase line here = y-axis.

Recipe for phase line analysis

① Find all equilibrium solutions of an auto. eq. by solving $g(y) = 0$.
Then plot them on the phase line.

$$\xrightarrow[\quad -1 \qquad\qquad 1 \qquad\qquad 2 \quad]{} y$$

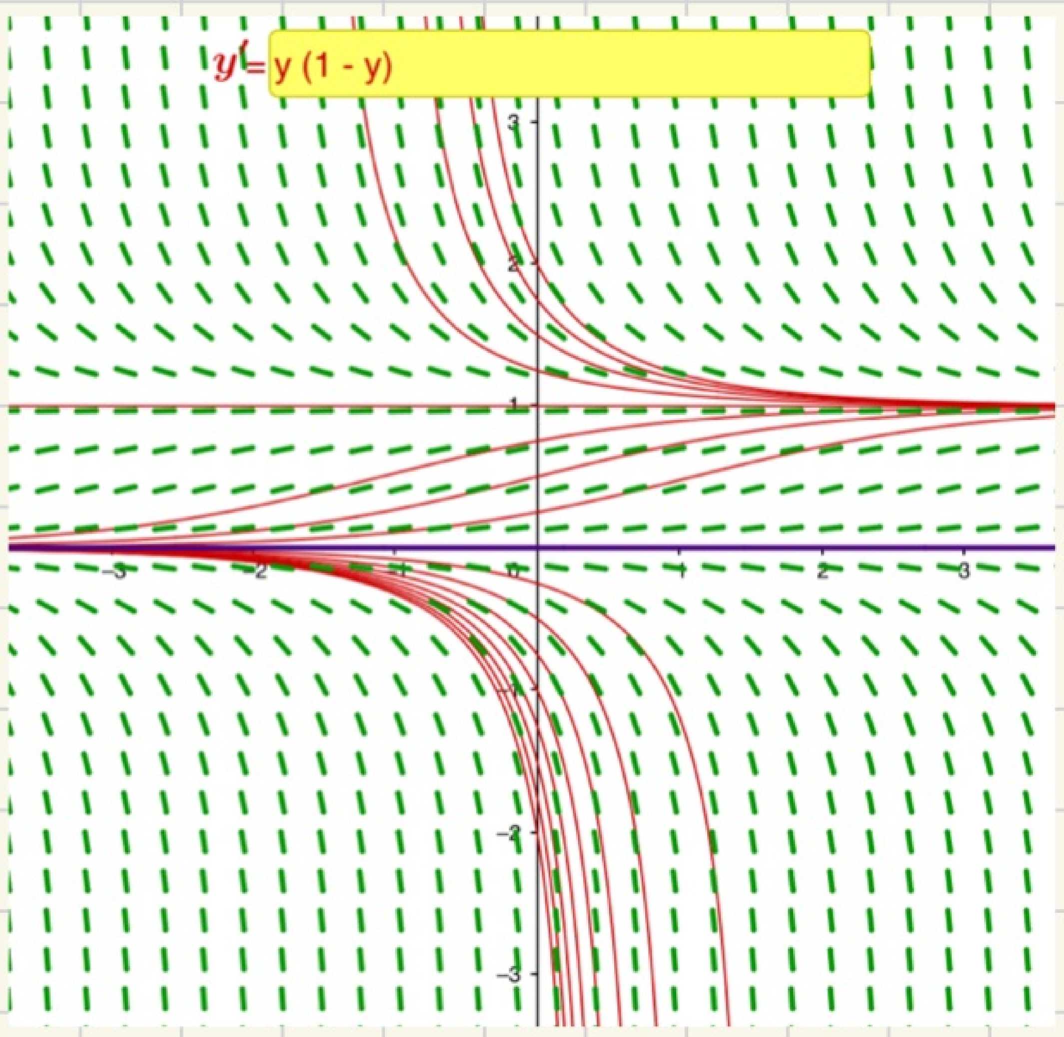
② These equilibrium pts devide phase line into serveral intervals, on each of which $g(y)$ should not change sign ($g(y)$ is contin. func.)
Then study sign of $g(y)$ on each interval $\Rightarrow$ sign of $\frac{dy}{dx}$ $\Rightarrow$ direction of solution of practical (solution) as $x \uparrow$.

③ Draw an arrow on each interval.

$y'>0$ → -1 ← $y'<0$ 1 $y'>0$ → 2 ← $y'<0$ y

④ If y_0 belongs to a $y'<0$ interval, $y(x) \downarrow$ to the edge of the interval as $x \to \infty$.
$y'>0$ $\uparrow$

e.g.



$\frac{dy}{dx} = y(1-y)$. Let $y(1-y) = 0 \Rightarrow y = 0$ or $y = 1$.

$-$ ← 0 → $+$ 1 ← $-$ y
$y'<0$ $y'>0$ $y'<0$

↳ $y_0 > 1$, $y(x) \downarrow$ to $y = 1$ as $x \to \infty$.
$y(x) \uparrow$ to ∞ as $x \to -\infty$.
$y_0 \in (0,1)$, $y(x) \uparrow$ to $y = 1$ as $x \to \infty$.
$y(x) \downarrow$ to $y = 0$ as $x \to -\infty$.
$y_0 < 0$, $y(x) \downarrow$ to $-\infty$ as $x \to \infty$.
$y(x) \uparrow$ to $y = 1$ as $x \to -\infty$.